1 NB

- ODE $\equiv$ Ordinary Differential Equation

2 Problems

Check the supplied information with Wikipedia as I might have made a misprint (and/or find there the missing information).

1. **Natural logarithm (integral representation).** Implement a function which calculates the natural logarithm of a real positive number x using the integral representation

$$\ln(x) = \int_1^x \frac{1}{t} dt \,. \tag{1}$$

   Employ an integration routine from GSL. Before calling the integration routine reduce the logarithm of an arbitrary positive number to the logarithm of a number in the range $1 \le x < 2$ using the formulae

$$\ln\left(\frac{1}{x}\right) = -\ln(x)\,, \tag{2}$$

$$\ln(x^2) = 2\ln(x)\,. \tag{3}$$

2. **Natural logarithm (ODE representation).** Do the same as in "Natural logarithm (integral representation)" but use the ODE representation,

$$\frac{dy}{dx} = \frac{1}{x}\,,\; y(1) = 0\,, \tag{4}$$

   and one of the ODE routines from GSL.

3. **Square root (ODE representation).** Implement a function which calculates the square root of a real positive number x using the ODE representation

$$\frac{dy}{dx} = \frac{1}{2y}\,,\; y(1) = 1\,. \tag{5}$$

   Employ an ODE routine from GSL. Before calling the ODE routine reduce the function of an arbitrary positive number to the function of a number in the range $1 \le x < 2$ using the formulae

$$\sqrt{\frac{1}{x}} = \frac{1}{\sqrt{x}}\,, \tag{6}$$

$$\sqrt{2x} = \sqrt{2}\sqrt{x}\,. \tag{7}$$

4. **Square root (root finding).** Implement a function which calculates the square root r of a real positive number x by solving the equation

$$r^2 - x = 0\,. \tag{8}$$

   Reduce the argument to $1 \le x < 2$ (as in "Square root (ODE representation)") and use the initial guess of 1. Use one of the GSL's root finding routines.

5. **Exponential function (ODE representation).** Implement the exponential function of a real number x using the ODE representation

$$\frac{dy}{dx} = y\,,\; y(0) = 1\,. \tag{9}$$

   Reduce the argument to $0 \le x < 1$ using the formulae

$$\exp(-x) = \frac{1}{\exp(x)}\,, \tag{10}$$

$$\exp(x) = \exp\left(\frac{x}{2}\right)^2 \tag{11}$$

and employ one the GSL's ODE routines.

6. **Sine function (ODE representation).** Implement the sine function of a real number x using the ODE representation
$$y'' = -y\,,\ y(0) = 0\,,\ y'(0) = 1\,. \tag{12}$$
Reduce the argument to, say, $0 \le x \le 2\pi$ using the periodicity and the symmetry of the sine function (and perhaps even further down to $0 \le x \le \pi/2$ using trigonometric identities) and then call one of the GSL's ODE routines.

7. **Cosine function (ODE representation).** Implement the cosine function of a real number x using the ODE representation
$$y'' = -y\,,\ y(0) = 1\,,\ y'(0) = 0\,. \tag{13}$$
Reduce the argument to, say, $0 \le x \le 2\pi$ using the periodicity and the symmetry of the cosine function (and perhaps even further down to $0 \le x \le \pi/2$ using trigonometric identities) and then call one of the GSL's ODE routines.

8. **Tangent function (ODE representation).** Implement the tangent function of a real number x using the ODE representation
$$y' = 1 + y^2\,,\ y(0) = 0\,. \tag{14}$$
Reduce the argument to, say, $0 \le x < \pi/2$ using the periodicity and the symmetry of the tangent function and then call one of the GSL's ODE routines.

9. **Implement the Arcsine function using integral representation.**
$$\arcsin(x) = \int_0^x \frac{1}{\sqrt{1-z^2}}\,dz, \qquad |x| \le 1\,. \tag{15}$$

10. **Implement the Arccosine function using integral representation.**
$$\arccos(x) = \int_x^1 \frac{1}{\sqrt{1-z^2}}\,dz, \qquad |x| \le 1\,. \tag{16}$$

11. **Implement the Arctangent function using inegral representation.**
$$\arctan(x) = \int_0^x \frac{1}{z^2+1}\,dz\,. \tag{17}$$

12. **Implement the Arctangent function using ODE representation.**
$$\arctan(x)' = \frac{1}{x^2+1},\ \arctan(0) = 0\,. \tag{18}$$

13. **Implement the Arccotangent function using inegral representation.**
$$\mathrm{arccot}(x) = \int_x^\infty \frac{1}{z^2+1}\,dz\,. \tag{19}$$

14. **Implement the Arccotangent function using ODE representation.**
$$\mathrm{arccot}(x)' = \frac{-1}{x^2+1},\ \mathrm{arccot}(0) = \frac{\pi}{2}\,. \tag{20}$$

15. **Implement the Bessel function of the first kind of integer index using Bessel's integral representation.**
$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos\left(nt - x\sin(t)\right)dt\,. \tag{21}$$

16. **Implement the Logistic function using ODE representation.**

$$y' = (1-y)y\,,\ y(0) = \frac{1}{2}\,. \tag{22}$$

17. **Diagonalisation of random real symmetric matrices.** Implement a function which takes a positive integer number n as the argument, generates an $n \times n$ random real symmetric matrix, diagonalizes it, and returns the largest eigenvalue as the result. Make a plot of the function (e.g. for $1 \leq n \leq 50$) and a reasonable fit. You can generate (pseudo)random real numbers uniformly distributed between zero and one using the `rand` function from `stdlib`,

    ```
    #define RND (double)rand()/RAND_MAX                    (23)
    ```

    You can sort eigenvalues with `gsl_sort_vector` function.

18. **Diagonalisation of projection matrices.** Implement a function which takes a positive integer number n as the argument, generates an $n \times n$ real symmetric matrix where all matrix elements are equal one, diagonalizes the matrix, and returns the largest eigenvalue. Make a plot of the function and explain the result. You can sort eigenvalues with `gsl_sort_vector` function.