

1 Ordinary least squares

1.1 Introduction

A system of linear equations is considered *overdetermined* if there are more equations than unknown variables. If all equations of an overdetermined system are linearly independent, the system has no exact solution.

A *linear least-squares problem* is the problem of finding an approximate solution to an overdetermined linear system. It often arises in applications where a theoretical model is fitted to experimental data.

1.2 Linear least-squares problem

Consider a linear system

$$\mathbf{A}\mathbf{c} = \mathbf{b}, \quad (1)$$

where $\mathbf{A}$ is an $n \times m$ matrix, $\mathbf{c}$ is an m -component vector of unknown variables and $\mathbf{b}$ is an n -component vector of the right-hand side terms. If the number of equations n is larger than the number of unknowns m , the system is overdetermined and generally has no solution.

However, it is still possible to find an approximate solution—the one where $\mathbf{A}\mathbf{c}$ is only approximately equal $\mathbf{b}$ —in the sense that the Euclidean norm of the difference between $\mathbf{A}\mathbf{c}$ and $\mathbf{b}$ is minimized,

$$\mathbf{c} : \min_{\mathbf{c}} \|\mathbf{A}\mathbf{c} - \mathbf{b}\|^2. \quad (2)$$

The problem (2) is called the ordinary least-squares problem and the vector $\mathbf{c}$ that minimizes $\|\mathbf{A}\mathbf{c} - \mathbf{b}\|^2$ is called the *least-squares solution*.

1.3 Solution via QR-decomposition

The linear least-squares problem can be solved by QR-decomposition. The matrix $\mathbf{A}$ is factorized as $\mathbf{A} = \mathbf{Q}\mathbf{R}$, where $\mathbf{Q}$ is $n \times m$ matrix with orthogonal columns, $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$, and $\mathbf{R}$ is an $m \times m$ upper triangular matrix. The Euclidean norm $\|\mathbf{A}\mathbf{c} - \mathbf{b}\|^2$ can then be rewritten as

$$\begin{aligned} \|\mathbf{A}\mathbf{c} - \mathbf{b}\|^2 &= \|\mathbf{Q}\mathbf{R}\mathbf{c} - \mathbf{b}\|^2 \\ &= \|\mathbf{R}\mathbf{c} - \mathbf{Q}^T\mathbf{b}\|^2 + \|(1 - \mathbf{Q}\mathbf{Q}^T)\mathbf{b}\|^2 \geq \|(1 - \mathbf{Q}\mathbf{Q}^T)\mathbf{b}\|^2. \end{aligned} \quad (3)$$

The term $\|(1 - \mathbf{Q}\mathbf{Q}^T)\mathbf{b}\|^2$ is independent of the variables $\mathbf{c}$ and can not be reduced by their variations. However, the term $\|\mathbf{R}\mathbf{c} - \mathbf{Q}^T\mathbf{b}\|^2$ can be reduced

down to zero by solving the $m \times m$ system of linear equations

$$\mathbf{R}\mathbf{c} = \mathbf{Q}^T\mathbf{b} . \quad (4)$$

The system is right-triangular and can be readily solved by back-substitution. Thus the solution to the ordinary least-squares problem (2) is given by the solution of the triangular system (4).

1.4 Ordinary least-squares curve fitting

Ordinary least-squares curve fitting is a problem of fitting n (experimental) data points $\{x_i, y_i \pm \Delta y_i\}_{i=1, \dots, n}$, where Δy_i are experimental errors, by a linear combination, $F_{\mathbf{c}}$, of m functions $\{f_k(x)\}_{k=1, \dots, m}$,

$$F_{\mathbf{c}}(x) = \sum_{k=1}^m c_k f_k(x) , \quad (5)$$

where the coefficients c_k are the fitting parameters.

The objective of the least-squares fit is to minimize the square deviation, called χ^2 , between the fitting function $F_{\mathbf{c}}(x)$ and the experimental data,

$$\chi^2 = \sum_{i=1}^n \left(\frac{F(x_i) - y_i}{\Delta y_i} \right)^2 . \quad (6)$$

where the individual deviations from experimental points are weighted with their inverse errors in order to promote contributions from the more precise measurements.

Minimization of χ^2 with respect to the coefficient c_k in (5) is apparently equivalent to the least-squares problem (2) where

$$A_{ik} = \frac{f_k(x_i)}{\Delta y_i} , \quad b_i = \frac{y_i}{\Delta y_i} . \quad (7)$$

If $\mathbf{QR} = \mathbf{A}$ is the QR-decomposition of the matrix $\mathbf{A}$, the formal least-squares solution to the fitting problem is

$$\mathbf{c} = \mathbf{R}^{-1}\mathbf{Q}^T\mathbf{b} . \quad (8)$$

In practice of course one rather back-substitutes the right-triangular system $\mathbf{R}\mathbf{c} = \mathbf{Q}^T\mathbf{b}$.

1.4.1 Variances and correlations of fitting parameters

Suppose δy_i is a small deviation of the measured value of the physical observable at hand from its exact value. The corresponding deviation δc_k of the fitting coefficient is then given as

$$\delta c_k = \sum_i \frac{\partial c_k}{\partial y_i} \delta y_i . \quad (9)$$

In a good experiment the deviations δy_i are statistically independent and distributed normally with the standard deviations Δy_i . The deviations (9) are then also distributed normally with *variances*

$$\langle \delta c_k \delta c_k \rangle = \sum_i \left(\frac{\partial c_k}{\partial y_i} \Delta y_i \right)^2 = \sum_i \left(\frac{\partial c_k}{\partial b_i} \right)^2 . \quad (10)$$

The standard errors in the fitting coefficients are then given as the square roots of variances,

$$\Delta c_k = \sqrt{\langle \delta c_k \delta c_k \rangle} = \sqrt{\sum_i \left(\frac{\partial c_k}{\partial b_i} \right)^2} . \quad (11)$$

The variances are diagonal elements of the *covariance matrix*, Σ , made of *covariances*,

$$\Sigma_{kq} \equiv \langle \delta c_k \delta c_q \rangle = \sum_i \frac{\partial c_k}{\partial b_i} \frac{\partial c_q}{\partial b_i} . \quad (12)$$

Covariances $\langle \delta c_k \delta c_q \rangle$ are measures of to what extent the coefficients c_k and c_q change together if the measured values y_i are varied. The normalized covariances,

$$\frac{\langle \delta c_k \delta c_q \rangle}{\sqrt{\langle \delta c_k \delta c_k \rangle \langle \delta c_q \delta c_q \rangle}} \quad (13)$$

are called *correlations*.

Using (12) and (8) the covariance matrix can be calculated as

$$\Sigma = \left(\frac{\partial \mathbf{c}}{\partial \mathbf{b}} \right) \left(\frac{\partial \mathbf{c}}{\partial \mathbf{b}} \right)^T = R^{-1} (R^{-1})^T = (R^T R)^{-1} = (A^T A)^{-1} . \quad (14)$$

The square roots of the diagonal elements of this matrix provide the estimates of the errors $\Delta \mathbf{c}$ of the fitting coefficients,

$$\Delta c_k = \sqrt{\Sigma_{kk}} \Big|_{k=1 \dots m} , \quad (15)$$

and the (normalized) off-diagonal elements provide the estimates of their correlations.

Here is a C-implementation of the ordinary least squares fit via QR decomposition,

```
void lsfit(vector* x, vector* y, vector* dy,
int nf, float f(int k, float x), vector* c, matrix* S){
    int n=x->size, m=nf;
    matrix *A = matrix_alloc(n,m), *R = matrix_alloc(m,m);
    vector *b = vector_copy_alloc(y); vector_div(b,dy);
    for(int i=0;i<n;i++){
        float xi=vector_get(x,i), dyi=vector_get(dy,i);
        for(int k=0;k<m;k++) matrix_set(A,i,k,f(k,xi)/dyi);
    }
    givens_qr(A); givens_qr_unpack_R(A,R);
    givens_qr_solve_inplace(A,b);
    for(int i=0;i<m;i++)vector_set(c,i,vector_get(b,i));
    matrix* Rinv=matrix_alloc(m,m);
    givens_qr_inverse(R,Rinv); matrixT_times_matrix(Rinv,Rinv,S);
    matrix_free(A); matrix_free(R); matrix_free(Rinv); vector_free(b);
}
```

An illustration of a fit is shown on Figure 1 where a polynomial is fitted to a set of data.

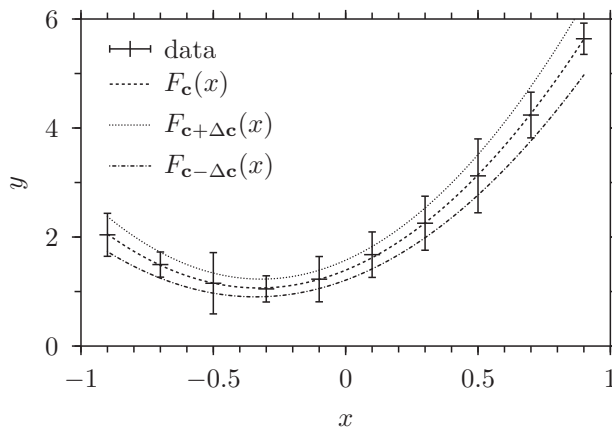


Figure 1: Ordinary least squares fit of $F_{\mathbf{c}}(x) = c_1 + c_2x + c_3x^2$ to a set of data. Shown are fits with optimal coefficients $\mathbf{c}$ as well as with $\mathbf{c} + \Delta\mathbf{c}$ and $\mathbf{c} - \Delta\mathbf{c}$.