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Bachelor Project

Gravitational Waves in Special and General Relativity

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15th January 2017

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Abstract

Gravitational waves are ripples in the curvature of spacetime, emitted by the most extreme and highly energetic processes in the Universe. Propagating outwards from the source, a gravitational wave warps the fabric of spacetime and distorts everything with the effect of strain. Gravitational radiation has clear similarities to electromagnetic radiation. In the weak field limit, gravitational waves may be described using a gravitoelectromagnetic analogy, where a gravitoelectric and a gravitomagnetic field make up the basis for equations analogue to Maxwell's field equations. Although the gravitoelectromagnetic analogy describes the concepts of gravitational waves well, it only substantiates the existence of gravitational waves. Einstein's theory of general relativity is more convincing. The Einstein field equations are a set of non-linear equations, describing the interactions between matter and curved spacetime, and thereby express the spacetime geometry. The non-linearity of the Einstein equations makes them difficult to solve, and approximative solutions of linear gravity are used instead, for weak gravitational waves in nearly flat spacetime. Here, gravitational waves are small ripples in spacetime, and the geometry may be described as a small perturbation of a flat spacetime metric. Although Einstein predicted gravitational waves in 1916, no evidence for their existence was found before 1979, when Hulse and Taylor found, that energy loss in a binary pulsar was in complete agreement with the expected emission of gravitational radiation. The laser interferometer gravitational wave observatory *LIGO*, made the first direct detection of a gravitational wave in 2015; an extremely important achievement, making an entirely new realm of information available to scientists. The detection of gravitational waves creates a new field of enquiry, and as technology improves in the years to come, so do the chances of making new and exciting discoveries.

Acknowledgements

I would like to send a very special and heartfelt thank you to my supervisor Ulrik, who despite being extremely busy, always found the time to support, advice, inspire and drink coffee. I am also grateful to Emma for valuable comments, and my dad for helping me proofread the project - despite declaring it to be "wickedness on paper".

1 Introduction

Gravitational waves are ripples in the fabric of spacetime. Emitted from the most extreme and highly energetic processes in the Universe, gravitational radiation carries not only energy, but also valuable information, away from a radiating system. Although they were predicted by Albert Einstein in 1916, the existence of gravitational waves was not confirmed, until they were indirectly measured in 1979 [15], and then successfully detected by *LIGO* in 2015 [24]. The detection of gravitational waves is an extremely important achievement, because it makes an entirely new realm of information available to scientists.

Massive accelerating bodies; such as supernova, rotating non-symmetrical stars and binary systems, emit quadrupole gravitational radiation. Gravitational radiation has some similarities to electromagnetic radiation, but is a completely different phenomenon. Unlike electromagnetic radiation, the interaction between gravitational radiation and matter is very weak, and a gravitational wave propagates unrestricted through spacetime as a result. This makes gravitational radiation extremely useful, because a detected signal is free of the warping and alterations electromagnetic waves undergo, when they travel through space. Gravitational radiation also makes it possible to collect information about astrophysical bodies, which emit little or no radiation at all, such as black holes [1].

This project examines some of the fundamental concepts of special and general relativity, and how gravitational waves are described using these theories. The aim is to get an extensive understanding of gravitational radiation, by considering both the gravitoelectromagnetic analogy and the more conventional approach by use of Einstein's field equations. Methods of detecting gravitational waves, both indirectly and directly, are also considered, and a simulated signal is compared to the very first gravitational wave detection. On a final note, it is considered, what might be expected from the detection of gravitational radiation in the years to come.

2 Gravitational Waves in Special Relativity

The existence of gravitational waves was predicted by Albert Einstein in 1916, when he published his paper *Approximative Integration of the Field Equations of Gravitation* [2].
However, using only the theory of special relativity and an analogy between gravity and electromagnetism, gravitational waves also appear plausible [3]. This slightly unconventional approach is based on the work of Oliver Heaviside, who in 1893 published a book on electromagnetic theory, containing the appendix *A Gravitational and Electromagnetic Analogy* [4]. Here, Heaviside suggested that the nature of gravity might be very similar to that of electromagnetism. The analogy is also referred to as gravitoelectromagnetism, and is a formal analogy between Maxwell's field equations (24) and Einsteins field equations (61) for general relativity, valid in the weak field limit.

2.1 Basics of Special Relativity

2.1.1 Four-vectors

To describe the geometry of spacetime, that is placement of points in time and space, four-vectors are used to express coordinates. A four-vector is defined as a directional line segment in

flat spacetime, similarly to a three-dimensional vector in a three dimensional Euclidean space. Spacetime four-vectors have the form,

$$\boldsymbol{x} = (x_0, x_1, x_2, x_4) \tag{1}$$

where the first dimension $x_0 = ct$ corresponds to the time-dimension, and the last three dimensions are the spatial dimensions of a point. Four-vectors may be multiplied, added and subtracted according to the usual rules of vectors. A point in spacetime may also be referred to as an *event*, since it occurs at a particular place and time [5, 5.1]. To distinguish four-vectors from the regular three-dimensional vectors, four-vectors are denoted by a bold symbol $\boldsymbol{x}$, while three-dimensional vectors are denoted by a vector arrow $\vec{x}$.

The components of a four-vector may vary in different inertial frames, because the coordinate four-vector basis is different. However, the length of a four-vector is invariant; it is the same in every inertial frame. This makes four-vectors extremely useful, since a law of physics described in four-vectors will take the same form in every inertial frame. The interval of the four-vector squared is given by:

$$ds^2 = -x_0^2 + x_1^2 + x_2^2 + x_3^2 \tag{2}$$

This interval may be either positive, negative or null, resulting in spacetime intervals that are timelike, spacelike and lightlike, respectively. The sign of the interval, depends on whether the absolute value of the sum of the spatial coordinates is larger than the time coordinate. Light propagates along lightlike intervals, where there is exactly the same amount of space and time [6, 2].

2.1.2 Inertial Frame and Lorentz Transformation

In special relativity, only observers moving uniformly at a constant velocity are of interest. Observers accelerating, either by changing their speed or direction, result in the far more complicated mathematics of general relativity. Special relativity is the special case of general relativity, where there is no acceleration [7, 1.1].

A frame of reference is a rectangular coordinate system moving with an observer, such that the coordinates are constant in time. When two observers are stationary relative to each other, they share a frame of reference. However, if one observer is moving uniformly relative to another observer, their reference frames are no longer equivalent. Special relativity aims at finding the relation between coordinates in two separate reference frames. Observers of any frame are free to set their frame to be the standard of rest, which means that there is no longer such a thing as absolute rest or motion. The laws of physics are not the same in all frames of reference, and only uniformly moving frames obey the law of inertia. These frames are called inertial frames [7, 1.3].

The relation between coordinates in two inertial frames of reference, is the symmetrical and linear Lorentz transformation, which expressed in four-vectors is:

$$\begin{array}{rclrcl} x_0' & = & \gamma\left(x_0 - x_1\frac{V}{c}\right), & x_2' & = & x_2 \\ x_1' & = & \gamma\left(x_1 - x_0\frac{V}{c}\right), & x_3' & = & x_3 \end{array} \tag{3}$$

where $\gamma = 1/\sqrt{1-\frac{V^2}{c^2}}$ is the Lorentz factor [8, 11.1.4]. Here the coordinates frames $S = (x_0, x_1, x_2, x_3)$ and $S' = (x_0', x_1', x_2', x_3')$ are moving uniformly with velocity V relative to each

other. The transformation is symmetrical, so if the sign of V is changed and the primed and unprimed coordinates are switched, the inverted transformation is found. Furthermore, the transformation law approaches the Galilean form in the non-relativistic limit [7, 4].

2.2 Gravitational Waves in Special Relativity

Before considering the gravitoelectromagnetic analogy, the well known gravitoelectric field $\vec{g}$ is assessed, and subsequently the necessity of a gravitomagnetic field $\vec{h}$ must be considered, as it is done by [9] and [3, 28].

2.2.1 The Gravitoelectric Field

Isaac Newton's classical law of universal gravity, describes the gravitational force between two bodies. The static gravitational force a body with mass M exerts on a mass m is,

$$\vec{F}_m = -G\frac{mM}{r^2}\hat{r}, \tag{4}$$

where G is the gravitational constant, r is the distance between the bodies' centre of mass, and $\hat{r} = \frac{\vec{r}}{r}$ is the unit vector directed from M to m. This force may also be expressed as $\vec{F}_m = m\vec{g}$, where

$$\vec{g} = -G\frac{M}{r^2}\hat{r}, \tag{5}$$

is the gravitational field created by mass M. It is noted, that the gravitational force (4) has a great resemblance to Coulombs law,

$$\vec{F}_q = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\hat{r}, \tag{6}$$

where ϵ_0 is the vacuum permittivity, the masses have been replaced by the charges Q and q, and the gravitational constant $-G$ has been replaced by $\frac{1}{4\pi\epsilon_0}$. It is noteworthy, that charged particles may be either positive or negative, whereas masses have but one sign. Consequently, the gravitational field is similar to that of an electric Coulomb field for two oppositely charged particles. This result is important for the multipole emission of gravitational waves, as it will be discussed in section 3.6.

The gravitational force in equation (4) is usually calculated for the case of two spherically symmetric bodies, but will now be applied to a case analogue to example 2.2 in an *Introduction to Electrodynamics* [8].
Suppose that the mass M is evenly distributed along a wire of length $\Delta x = 2L$, and that the point mass m is placed at a distance y perpendicular to the centre of the wire, as seen in figure (1). The linear mass density of the wire is $\lambda = \frac{M}{2L}$, and the mass of a segment dx of the wire, is thus

$$\lambda dx = \frac{M dx}{2L}.$$

If the axis is defined such that $x = 0$ at the centre of the wire, the x-components of the gravitational force cancel each other out, and it is therefore enough to consider the y-component,

$$\vec{F} = -2Gm\frac{\lambda dx}{r^2}\cos\theta\hat{y}, \tag{7}$$

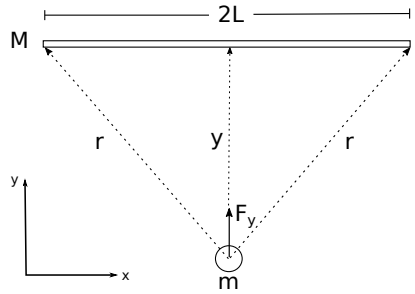

Figure 1: Point mass m is placed a distance y from the centre of a wire with length $2L$ and mass M.

where $\cos\theta = \frac{y}{r}$ and $r = \sqrt{x^2+y^2}$. Here, the distance y is a constant value, while the x-coordinate runs over the interval $[0; L]$. Using this information it is now possible to calculate the magnitude of the force by integrating over the interval of x,

$$\begin{aligned} F &= 2Gm\int_0^L \frac{\lambda}{x^2+y^2}\frac{y}{\sqrt{x^2+y^2}}dx \\ &= 2Gm\frac{\lambda L}{y\sqrt{L^2+y^2}} \end{aligned} \tag{8}$$

This expression is in agreement with equation (4), when the distance becomes $y \gg L$, and the wire may be approximated as a point mass.
Analogue to the electromagnetic example, the gravitational field of an infinite straight wire is,

$$\vec{g} = -G\frac{2\lambda}{y}\hat{y}, \tag{9}$$

This striking similarity between the electric and gravitational field, and the fact that magnetic fields are created by moving charges, suggests the plausibility of a gravitomagnetic field for moving masses. Inspired by the fact, that a magnetic field may be considered as a relativistic transformation of an electric field [8, chapter 11.3.2], the moving gravitoelectric field is now considered.

2.2.2 The Gravitomagnetic Field

It is now necessary to consider the established setup from another frame of reference, and the system is therefore said to be in inertial frame S'. The wire is parallel to the x'-axis, and the point mass m is placed in the $x'y'$-plane a distance $-y'$ from the wire, as it is seen in figure 2. According to equation (9), the gravitational force exerted on m is given as,

$$\vec{F'} = \left(0, \frac{2Gm\lambda}{y'}, 0\right). \tag{10}$$

The mass m is now set to move in S' with velocity $\vec{v} = (0, v'_y, 0)$ towards the wire.
It is possible to make a transformation to the inertial frame S, in which the wire moves with velocity $\vec{v} = (v_x, 0, 0)$; that is, the inertial system S' moves at velocity $V = v_x$ relative to S as seen in figure 3.
The aim is to transform $\vec{F'}$ to the inertial system S. This is most easily accomplished by expressing the force as a four-force, which is written as,

$$\boldsymbol{K}' = \gamma'\left(\frac{\vec{F'}\cdot\vec{v'}}{c}, \vec{F'}\right) = \gamma'\left(\frac{F'_y v'_y}{c}, 0, F'_y, 0\right), \tag{11}$$

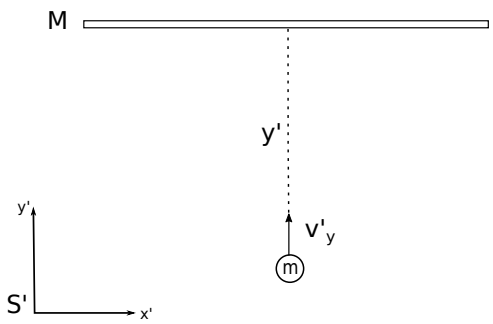

Figure 2: System seen from rest frame of reference S'.

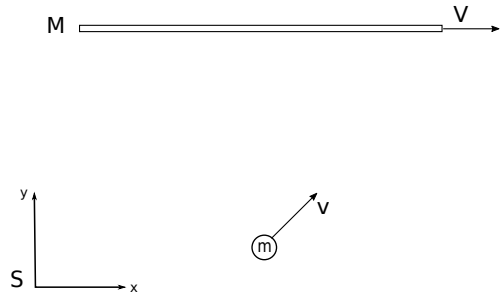

Figure 3: System seen from a frame of reference S, where the wire moves with a velocity $\vec{V}$.

where $\gamma' = \gamma(v') = \frac{1}{\sqrt{1-v'^2/c^2}}$ is the Lorentz factor, $\vec{F'} = (F'_x, F'_y, F'_z)$ is the force vector (10), the velocity is $\vec{v'} = (v'_x, v'_y, v'_z)$, and c is the speed of light. The components of the four-vector transforms according to the Lorentz transformation (3):

$$\begin{aligned} K_0 &= \Gamma\left(K'_0 + K'_x \tfrac{V}{c}\right), \quad K_y = K'_y \\ K_x &= \Gamma\left(K'_x + K'_0 \tfrac{V}{c}\right), \quad K_z = K'_z \end{aligned} \tag{12}$$

where $\Gamma = \gamma(V)$. The mass has a Lorentz factor γ' in the system of inertia S', and then the factor will transform to the factor $\gamma = \gamma(v)$ in S, with the relation,

$$\gamma = \left(1 + \frac{v'_x V}{c^2}\right)\Gamma\gamma' = \Gamma\gamma' \tag{13}$$

where $v'_x = 0$ since the wire is stationary in this frame [3, eq. 10.58].
The z-component of the gravitational force remains unchanged according to (12), since $\gamma' F'_z = \gamma F_z = 0$. Meanwhile, since $K'_y = K_y$, the y-component of the gravitational force is,

$$F_y = \frac{\gamma'}{\gamma} F'_y = \frac{F'_y}{\Gamma}. \tag{14}$$

The last step follows from the transformation of the Lorentz factor (13). The x-component is the final component of the force, which using to the Lorentz transformation (12), becomes

$$\begin{aligned} K_x &= \Gamma\left(K'_x + K'_0 \frac{V}{c}\right) = \Gamma \frac{F'_y \gamma' v'_y V}{c^2}, \\ F_x^m &= \frac{K_x}{\gamma} = F'_y \frac{v'_y V}{c^2} = \frac{2Gm\lambda}{y}\Gamma\frac{v_y V}{c^2} \end{aligned} \tag{15}$$

since $v'_y = \gamma v_y$ and $K'_x = 0$. The x-component of the force has been labeled F_x^m, because it is a contribution to the gravitomagnetic field, which shall momentarily become clear. Thus, in addition to the force F_y on the particle due to the gravitational attraction by the wire, there is also a non-zero x-component parallel to the wire.

D. Bedford explains that the necessity of an additional force is easily understood, since the point mass m requires a force parallel to V, in order to keep it's velocity constant [9]. He also argues, that this force is the gravitational analogue of the magnetic force, as it shall be seen subsequently.

In the inertial system S, the linear mass density of the wire will increase by a factor Γ, as the wire undergoes length contraction. Using this information, the gravitoelectric force in equation (10), may in the system S be written as

$$F_y^e = \Gamma \frac{2Gm\lambda}{y} = \frac{F_y' \Gamma^2}{\Gamma}. \tag{16}$$

From this, the gravitational vector field of the system, has the form:

$$\vec{g} = \frac{2G\lambda\Gamma}{y}\hat{y}. \tag{17}$$

Since equation (14) must hold true, such that $\frac{F_y'}{\Gamma} = F_y^e + F_y^m$, the additional force of the y-component is found to be,

$$F_y^m = \frac{F_y'}{\Gamma}(1-\Gamma^2) = \frac{F_y'}{\Gamma}\left(\frac{1-\frac{V^2}{c^2}}{1-\frac{V^2}{c^2}} - \frac{1}{1-\frac{V^2}{c^2}}\right) \tag{18}$$

$$= -\frac{F_y'}{\Gamma}\Gamma^2\frac{V^2}{c^2} = -\frac{2Gm\lambda}{y}\frac{\Gamma v_x V}{c^2}. \tag{19}$$

Combining equations (15) and (19), the gravitomagnetic force is,

$$\begin{aligned} \vec{F}^m &= F_x^m + F_y^m + F_z^m \\ &= \frac{V}{c^2}\frac{2Gm\lambda\Gamma}{y}(v_y, -v_x, 0). \end{aligned} \tag{20}$$

From this, the gravitomagnetic field may be expressed as,

$$\vec{h} = \frac{V}{c^2}\frac{2G\lambda\Gamma}{y}\hat{z}. \tag{21}$$

The combination of the gravitomagnetic force and gravitoelectric forces show great resemblance to the electromagnetic Lorentz force,

Lorentz force:

$$\vec{F}_{EM} = q\vec{E} + q\vec{v_q} \times \vec{B} \tag{22}$$

Gravtitational force:

$$\vec{F}_G = m\vec{g} + m\vec{v} \times \vec{h} \tag{23}$$

The Lorentz force expresses the combination of electric and magnetic force on a point charge due to an electromagnetic field. If point masses in a gravitational field are affected by the gravitational forces analogue to that of point charges in an electromagnetic field, it seems reasonable to assume that other aspects of electromagnetism may be applied in the same manner. However, one should consider the variations caused by the lack of gravitational "charges", since gravitational force is always one of attraction - never repulsion. Therefore, the gravitomagnetic interaction will always resemble that of two opposite electric charges.

2.3 Gravtioelectromagnetic Analogy

Now that the analogy between the electric field $\vec{E}$ and the magnetic field $\vec{B}$ and the two components of the gravitational field $\vec{g}$ and $\vec{h}$ has been recognised, it is time to consider gravitational waves in an analogy to Maxwell's equations for electromagnetic waves.
Maxwell's equations for the electric and magnetic field have the form,

$$\begin{aligned} \nabla \cdot \vec{E} &= \tfrac{\rho}{\epsilon_0}, & \nabla \times \vec{E} &= -\tfrac{\partial \vec{B}}{\partial t}, \\ \nabla \cdot \vec{B} &= 0, & \nabla \times \vec{B} &= \mu_0 \left(\vec{J} + \epsilon_0 \tfrac{\partial \vec{E}}{\partial t} \right), \end{aligned} \tag{24}$$

where μ_0 is the permeability of free space, ϵ_0 is the permittivity of free space, ρ is the electric charge density, and $\vec{J} = \rho \vec{v}$ is the electric current density.
The analogue gravitational equations are,

$$\begin{aligned} \nabla \cdot \vec{g} &= -4\pi G \rho_m, & \nabla \times \vec{g} &= -\tfrac{\partial \vec{h}}{\partial t}, \\ \nabla \cdot \vec{h} &= 0, & \nabla \times \vec{h} &= -4\pi H \rho_m \vec{v} + \tfrac{H}{G} \tfrac{\partial \vec{g}}{\partial t}, \end{aligned} \tag{25}$$

where ρ_m is the mass density, G is the gravitational constant, and H is the Heaviside constant charactering the strength of the gravitomagnetic interactions [10]. It is noted that the equations of (25) have the same form as those of (24), but the constants ϵ_0 and μ_0 have been replaced with the constants $\frac{-1}{4\pi G}$ and $-4\pi H$, respectively.
Considering the equations in empty space with $\rho_m = 0$, the expressions of (25) are simplified to,

$$\begin{aligned} \nabla \cdot \vec{g} &= 0, & \nabla \times \vec{g} &= -\tfrac{\partial \vec{h}}{\partial t}, \\ \nabla \cdot \vec{h} &= 0, & \nabla \times \vec{h} &= \tfrac{H}{G} \tfrac{\partial \vec{g}}{\partial t}. \end{aligned} \tag{26}$$

For the simplified Maxwell's electromagnetic equations, it is possible to derive the wave equations by taking the curl of the field vectors. The solution to these equations are expressions for the electric and magnetic waves, and from these the propagation velocity of electromagnetic waves in vacuum, is found to be

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}. \tag{27}$$

The constants analogue to the vacuum permittivity and permeability have already been noted, and inserting these values in equation (27), allows for a propagation velocity,

$$v_g = \sqrt{\frac{G}{H}} \approx c. \tag{28}$$

Although Heaviside considered that gravitational waves could propagate with the speed of light, he did not make any measurements to realise the final approximation in the equation. An experimental determination of the Heaviside constant H is indeed challenging, considering that it has a value c^2 times smaller than the gravitational constant G.

One might find it strange, or question why gravitational waves travel at the speed of light, but Richard Hammond explained, that this feature is common for all fields of infinite range or massless quanta. The reason for this is, that all these fields have wave equations, to which the

wave solutions may travel at the speed of light [11]. It then seems very reasonable that both electromagnetic waves and gravitational waves should travel with the velocity c.

3 Gravitational Waves in General Relativity

Gravitational waves are now explored, as they are described in general relativity. In special relativity all calculations are valid because a flat spacetime is assumed. In general relativity however, the curvature of the four-dimensional spacetime is a key element, and gravity is considered to be the geometry of this curvature. Before dealing with gravitational waves as solutions for the Einstein equations, some basic concepts of general relativity must be examined.

Gravitational waves can be described using linearised gravity. This makes it possible to use the weak-field metric to find solutions to the Einstein equation in a spacetime with a geometry close to flat spacetime. Units of $c = 1$, $G = 1$ will be used throughout this chapter.

3.1 The Basics of Curved Spacetime

It is an experimentally verified fact that all bodies, independent of mass and composition, fall with the same acceleration in a uniform gravitational field. This concept has been known since 1589, where the italian physicist Galileo Galilei possibly made his famous experiment from the Leaning Tower of Pisa. Here, Galileo measured that all bodies have the same time of descent, independent of mass [12, 1.1].

Albert Einstein was the first to make the connection between the uniqueness of path in spacetime, and the geometry of this four-dimensional union of space and time. He proposed, that the presence of mass causes the geometry of spacetime around it to curve depending on the size of the mass. Furthermore, he also suggested that bodies free of forces follow the straightest possible path in this slightly non-Euclidian geometry, in the same way that Newton declared bodies to follow a straight line in flat spacetime. This would mean that the Earth orbits the Sun, not because a gravitational force is acting upon it, but because it follows the straightest possible path along the curved geometry around the Sun [5, 2.2].

3.1.1 The Equivalence Principle

The principle of equivalence is the idea, that gravity and acceleration result in forces indistinguishable from each other. This means, that if an observer is placed in an closed box and experiences a downwards force equal to $\vec{g}$, they cannot tell whether this force is caused by the Earth's gravitational field, or if the force is caused by an upwards acceleration $\vec{g}$ in empty space. The equivalence principle makes it impossible to talk of absolute acceleration of a frame of reference, since no experiment can distinguish between force and acceleration [7, 8.2]. It is important to note, that this weak equivalence principle only holds true locally; the gravitational field strength must be uniform across the area of inspection. If the gravitational field increases or decreases in strength over the freely falling box, the particles inside it would experience tidal forces if falling towards the Earth. Here, it would be possible to distinguish acceleration and gravity by the movement of particles or lack thereof, respectively. The equivalence principle has some physical consequences: the acceleration of a small test mass is independent on its mass and composition;

light rays are bend by the gravitational field of massive objects; and the empirically determined equality between inertial and gravitational mass is explained [13].

3.1.2 Four-vectors in Curved Spacetime

The definition of four-vectors as a directed line-segment, has to be modified. The key element is to recognise that all vectorial quantities are measured locally in curved spacetime. This means, that the concept of magnitude and direction must be separated, and that direction can only be defined locally by means of very small vectors. It is possible to build larger vectors, by the usual rules of flat spacetime. However, these vectors are not inside the fabric spacetime, but rather define a flat tangent space upon the curved spacetime geometry, as it is demonstrated in figure 4.

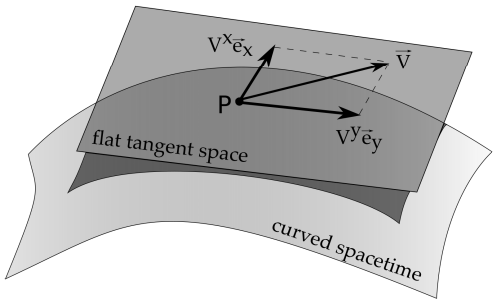

Figure 4: This idealised conceptual diagram demonstrates, how a velocity can be determined in a tangent space at a point P. Here the vector can be added, subtracted and multiplied by a scalar. However, vectors at different points are in different tangent spaces, and thus cannot simply be added. Adapted from [5, Figure 7.6]

This means that the idea of position vectors and displacement vectors must be discarded. Only displacement vectors between infinitesimally separated points have meaning.
A vector can be expressed as a linear combination of a basis four-vector $\boldsymbol{e_\mu}$, at the point x^μ,

$$\begin{aligned} \boldsymbol{a} &= a^\mu(x)\boldsymbol{e}_\mu(x), \\ &= a^0\boldsymbol{e}_0 + a^1\boldsymbol{e}_1 + a^2\boldsymbol{e}_2 + a^3\boldsymbol{e}_3, \end{aligned} \tag{29}$$

where $a^\mu(x)$ are components of the four-vector $\boldsymbol{a}$. Repeated Greek vector index is called Einstein summation convention and means, that the terms must be summarised from 0 to 3 [6, 1.2]. The scalar product of vectors in a coordinate basis is,

$$\boldsymbol{a} \cdot \boldsymbol{b} = g_{\mu\nu}a^\mu b^\nu. \tag{30}$$

Here, $g_{\mu\nu}$ is called a metric tensor and combined with the tangent vectors $a^\mu b^\nu$, it produces a real number in the same manner as a scalar product in Euclidean geometry.
In general relativity, an orthonormal basis consists of four mutually orthogonal vectors of unit length, where three are spacelike and one is timelike [5, 7.8].
Although the concept of vectors introduced here is physically accurate, it is mathematical imprecise. In curved spacetime, vectors are better defined as directional derivatives,

$$\boldsymbol{a} \equiv a^\mu \frac{\partial}{\partial x^\mu}. \tag{31}$$

Hereoff it follows that the partial derivative $\frac{\partial}{\partial x^\mu} = \partial_\mu$ are the coordinate basis vectors, and can be thought of as just another notation. The linear space of directional derivatives is the tangent

space [5, 20.1]. The equivalence principle suggests, that the characteristics of spacetime in special relativity hold true for local sections of curved spacetime; which is used when defining vectors on a tangent space. When it is possible to introduce new coordinates x'^{μ} to a metric $g'_{\mu\nu}(x'_P)$ at point x_P, such that

$$g_{\mu\nu}(x'_P) = \eta_{\mu\nu}, \quad \left.\frac{\partial g'_{\mu\nu}}{\partial x'^{\lambda}}\right|_{x=x_P} = 0, \tag{32}$$

where $\eta_{\mu\nu}$ is the flat spacetime metric; the infinitesimal neighbourhood around this point is a local inertial frame [5, 7.4].

3.1.3 Dual Vectors and Tensors

Tensors are ideal for expressing calculations in curved spacetime, since any calculation expressed in tensors, are satisfied in any basis, as it shall be seen subsequently.
A dual vector is a linear map from vectors to real numbers. Any vector $\boldsymbol{a}$ describes a linear map from another vector $\boldsymbol{b}$ to a real number, by

$$a(\boldsymbol{b}) \equiv \boldsymbol{a} \cdot \boldsymbol{b}. \tag{33}$$

This means, that for every vector there is a dual vector. The correspondence between the vector with coordinate basis components a^{μ}, and the dual vector with components a_{μ}, is given by

$$a_{\mu} = g_{\mu\nu}a^{\nu}, \quad a^{\mu} = g^{\mu\nu}a_{\nu}. \tag{34}$$

These relations are called lowering and raising an index, respectively. The connection can be inverted by introducing the matrix inverse of $g_{\mu\nu}$,

$$g^{\mu\lambda}g_{\lambda\nu} = \delta^{\mu}_{\nu} \tag{35}$$

Since there is no physical distinction between the quantity of a vector and a dual vector, and the mathematical correspondence is one-to-one, dual vectors can be thought of as components of vectors. The scalar product between two vectors can be written in a variety of ways:

$$\boldsymbol{a} \cdot \boldsymbol{b} = g_{\mu\nu}a^{\mu}b^{\nu} = a_{\mu}b^{\mu} = a^{\mu}b_{\mu} = g^{\mu\nu}a_{\mu}b_{\mu}. \tag{36}$$

The indices of vectors can be either upstairs or downstairs, related by the operation (34). Indices may be raised or lowered in an equation, as longs as free indices on both sides are balanced, and repeated indices always appear in upper-lower pairs [5, 20.2].

A tensor is very similar to a dual vector in the way, that a tensor is a linear map from *pairs* of vectors to real numbers. A tensor of rank n, is a linear map from n vectors into real numbers, and hence a vector is a tensor of first rank, since it consists of only one vector. The metric is a second-rank tensor. A vector $\boldsymbol{a}$ may be combined with a third-rank tensor $\boldsymbol{t}$, to form a second-rank tensor,

$$t_{\mu\nu\lambda}a^{\lambda}. \tag{37}$$

The simplest way of constructing a tensor is by taking the product of vectors, for example a third-rank tensor might be $u^{\mu}v^{\nu}w^{\lambda} = t^{\mu\nu\lambda}$.
Tensors can easily be converted from one basis to another. A tensor may be converted from a

coordinate basis to an orthonormal basis or another coordinate basis. Transforming the vector a'_ν from one coordinate basis x'^ν to the basis x^μ is done by,

$$a'_\nu = \frac{\partial x^\mu}{\partial x'^\nu} a_\mu. \tag{38}$$

The transformation of a tensor $t'_{\mu\nu}$ from one coordinate basis $x'_{\mu\nu}$ to another $x_{\lambda\sigma}$, is similar to (38) and has the general form:

$$t'_{\mu\nu}(x) = \frac{\partial x^\lambda}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} t'_{\lambda\sigma}(x) \tag{39}$$

Here, it is clear to see that if the component of a tensor is zero in one basis, it is zero in any other basis. This fact is useful when finding tensor relations, and may be used to show that relations between tensors hold in every basis.

3.1.4 Covariant Derivative

The covariant derivative is the derivative used for differentiating vectors, as it is explained in [5, 10.4]. The derivative of a vector involves the difference between the vector and nearby vectors at other points, which is rather problematic. A four-vector in curved space is defined at one specific point, and two nearby vectors cannot immediately be compared, because they are defined in separate tangent spaces. In order to define derivative of vectors, it is necessary to transport a vector from one spacetime point to another. To the first order of the displacement, the change in components will be a sum of two factors; the displaced position, and the change in basis vectors during transportation from one tangent space to another. The covariant derivative of the vector $\boldsymbol{v}$ in a coordinate basis turns out to be,

$$\nabla_\mu v^\nu = \frac{\partial v^\nu}{\partial x^\mu} + \Gamma^\nu_{\mu\lambda} v^\lambda \tag{40}$$

Here the coefficients on the form $\Gamma^\mu_{\nu\lambda}$ are called *Christoffel symbols*, and these are found from the metric and its derivatives,

$$g_{\mu\sigma}\Gamma^\sigma_{\nu\lambda} = \frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\mu\lambda}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu}\right). \tag{41}$$

The covariant derivative can also be generalised for tensors. The covariant derivative of a second-rank tensor is:

$$\nabla_\lambda t^\mu_\nu = \frac{\partial t^\mu_\nu}{\partial x^\lambda} - \Gamma^\mu_{\lambda\sigma} t^\sigma_\nu - \Gamma^\sigma_{\lambda\nu} t^\mu_\sigma. \tag{42}$$

In curved spacetime, vectors are not constant, when they have constant components. Instead, they are vector fields that do not change when transported in any direction.

3.1.5 Geodesics

A test mass is free, or freely falling, when it is under no other influence than the geometry of gravity. The test mass must have a mass so small, that it does not itself create any significant curvature in spacetime. A free test particle between two timelike separated points in curved spacetime, will extremize the proper time between them; here the proper time being the time measured by a clock travelling along the path. The extremal proper time world lines of test

particles are called geodesics, and the equations of motion is called the geodesic equation [5, 8.1]. The geodesic equation of a timelike geodesic is given by,

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}, \tag{43}$$

where τ is the proper time, that is, the time measured by the test mass. Equation (43) is really four equations of motion, one for each value of the free index μ. Alternatively, the geodesic equation could also be written in terms of the four velocity, $u^\mu = \frac{dx^\mu}{d\tau}$, such that,

$$\frac{du^\mu}{d\tau} = -\Gamma^\mu_{\nu\lambda}u^\nu u^\lambda. \tag{44}$$

Given an initial four velocity and location in spacetime, it is possible to find the values of these at a later moment in proper time.

The geodesic equation (43) is for timelike geodesics, but light rays and gravitational waves travel along null geodesics. Light rays move along null world lines for which $ds^2 = 0$. Null geodesics have the form,

$$\frac{d^2x^\mu}{d\rho^2} = -\Gamma^\mu_{\nu\lambda}\frac{dx^\mu}{d\rho}\frac{dx^\lambda}{d\rho}, \tag{45}$$

where the parameter ρ is chosen, such that the equation (45) takes the form of (43).

3.2 Linearised Gravitational Waves

In general relativity gravitational waves are most easily described when propagating in nearly flat spacetime, that is, in nearly empty space. Gravitational waves far from their source are small ripples in spacetime and cause small curvatures in the otherwise flat geometry. Gravitational waves in this approximation are called *linearised* gravitational waves.

In the weak field limit spacetime is close to a flat geometry. The metric $g_{\mu\nu}(x)$ describing the geometry of the approximately flat spacetime can be written as a perturbation of the flat spacetime metric,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \tag{46}$$

where $\eta_{\mu\nu}$ is the flat spacetime Minkowski metric:

$$\eta_{\mu\nu} = \begin{array}{c} \\ t \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} t & x & y & z \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \tag{47}$$

and $h_{\mu\nu}(x)$ is a small perturbation. The metric perturbation $h_{\mu\nu}(x)$ may describe the curvature of spacetime caused by a gravitational wave. The geometric metric $g_{\mu\nu}(x)$ for flat space in special relativity is the Minkowski metric (47), because there is no curvature and $h_{\mu\nu}(x) = 0$.

A linearised gravitational wave propagating along a z-axis, might have a perturbation element:

$$h_{\mu\nu}(t,z) = \begin{array}{c} \\ t \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} t & x & y & z \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} f(t-z) \tag{48}$$

where $f(t-z)$ is any function which satisfy $|f(t-z)| \ll 1$. The function $f(t-z)$ expresses the amplitude of the transverse components of the perturbation. For a plane gravitational wave propagating along the z-axis, the harmonic oscillating amplitude is only dependent on the time and distance from the source along the z-axis. The function $f(t-z)$ could be an exponential function or a sinusoidal function. A useful property of linearised plane gravitational waves is, that they can be added to make new solutions to the Einstein equations.
It is important to note, that the linearised gravitational waves do not give exact solutions to the Einstein equations. They are however, excellent approximations in the weak field limit, where the amplitude of the waves are small [5, 16.1].

3.2.1 Polarisation

Gravitational waves are transverse waves, so a wave travelling along the z-axis will cause spacetime to expand or contract in the x and y direction. The signature strain in spacetime caused by gravitational wave, is shown in figure (5).

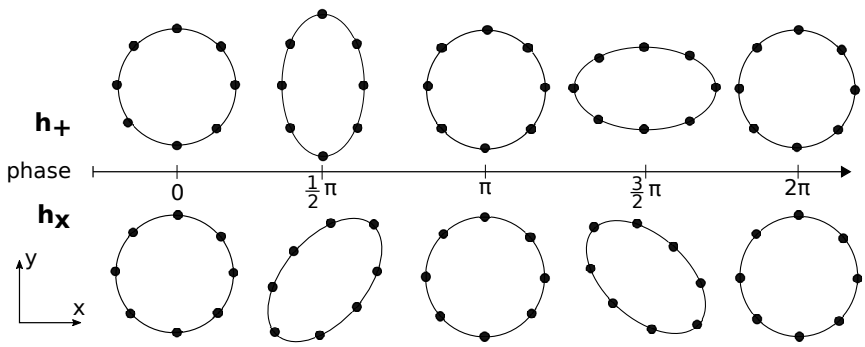

Figure 5: This figure shows the behaviour of 8 freely falling test masses arranged at an equal distance from the origin. The test masses are all in a plane perpendicular to the direction of the propagating gravitational wave, and show the warping of spacetime in the x and y directions, for $+$ and $\times$ polarisation. The coordinates of the test masses do not change, but the separations between them do vary.

The test masses reveal the warping of spacetime, as their coordinates remain constant, while the distance between them changes. For $+$-polarisation, spacetime is first squeezed along the x-axis and expanded along the y-axis, and then vice versa. The pattern is an ellipse, where the test masses oscillate periodically in time, but out of phase with each other. The other polarisation is labelled $\times$-polarisation, and this is the same pattern and sequence of displacement, only rotated 45°. This $\times$-polarisation is another solution to the Einstein equation in the (t, x, y, z) coordinates, and it has the form,

$$h_{\mu\nu}(t,z) = \begin{array}{c} \\ t \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} t & x & y & z \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} f(t-z) \tag{49}$$

It can be shown that $+$- and $\times$-polarisation are the only possible polarisations, and that these are independent of each other [6, 4.1]. This results in the most general form [5, 16.3] of the small

perturbation, of a linearised gravitational wave propagating in along the z-axis, is then:

$$h_{\mu\nu}(t,z) = \begin{array}{c} \\ t \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} t & x & y & z \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & f_+(t-z) & f_\times(t-z) & 0 \\ 0 & f_\times(t-z) & -f_+(t-z) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \tag{50}$$

where $f_+(t-z)$ is for plus polarisation, while $f_\times(t-z)$ is for cross polarisation of a gravitational wave propagating along the $z-$axis.

3.3 Curvature and the Einstein Equation

The presence of matter produces spacetime curvature, that is, a measure of local spacetime curvature is a measure of matter energy density. The equations relating matter and curvature are called the *Einstein equations*, and these equations are the field equations for general relativity, analogue to Maxwell's equations in electrodynamics. Einsteins equations are a set of 10 non-linear second-order partial differential equations for the metric coefficients $g_{\mu\nu}(x)$.

3.3.1 Riemann Curvature

To study the curvature of spacetime, the relative motion of at least two test masses is needed. If an observer has a world line $\boldsymbol{u}$ and a proper time τ and observes a test mass on another geodesic, the displacement vector between the two geodesics is $\boldsymbol{\chi}$, as it is described in [5, 21.1] and illustrated in figure 6.

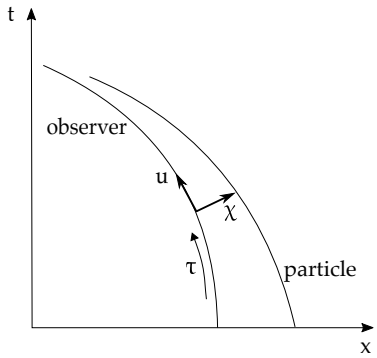

Figure 6: The world line of an observer and a nearby particle separated by the separation four-vector χ. Both freely falling, they move on geodesics; and the acceleration of the separation vector, along the observer's world line, is a quantitative measure for spacetime curvature. Adapted from [5, Figure 21.3]

The equation describing this motion is the equation of geodesic deviation, and this involves the separation four-vector $\boldsymbol{\chi}$ giving the infinitesimal displacement between two nearby geodesics.

$$(\boldsymbol{\nabla_u \nabla_u \chi})^\mu = -R^\mu_{\nu\lambda\sigma} u^\nu \chi^\lambda u^\sigma, \tag{51}$$

here $\boldsymbol{u}$ is the geodesic of an observer's world line and $\boldsymbol{\nabla_u \nabla_u \chi}$ is the squared covariant derivative (40), with respect to the proper time, of the separation vector along $\boldsymbol{u}$. The factor $R^\mu_{\nu\lambda\sigma}$ is called the *Riemann curvature tensor*. Equation (51) is the relativistic generalisation of the equation of

motion for the deviation between two particles.
In Newtonian gravity, the deviation equation is:

$$\frac{d^2\chi^i}{dt^2} = -\delta^{ij}\left(\frac{\partial^2\Phi}{\partial x^j \partial x^k}\right)\chi^k \tag{52}$$

where $\Phi(x)$ is the gravitational potential, and the separation vector $\vec{\chi}$ gives the spatial separation between two nearby particles. The second partial derivative, $\frac{\partial^2\Phi}{\partial x^j \partial x^k}$, gives the differential acceleration and thereby the forces which either pull the particles towards each other or push them apart. The length of the separation vector may vary in time. If a freely falling particle is placed at a radial distance $x^i(t)$ from the Earth, and a nearby particle is placed at a radial distance $x^i(t) + \chi^i(t)$, the length of the separation vector will increase with time. This is because the first particle at $x^i(t)$ experiences a greater force from the Earth's radial gravitational field, than the particle falling at the distance $x^i(t) + \chi^i(t)$. The forces acting on the particles are also called tidal forces. Given the separation of two nearby particles at one point in time, equation (52) can find the separation of the particles, as long as the separation vector $|\vec{\chi}|$ remains small.

The similar structure of the two deviation equations (51) and (52) reveals, that the Riemann curvature tensor is the measure of spacetime curvature. Analogue to how the gravitational potential determines how two test particles accelerate from or towards each other (52), the Riemann curvature tensor describes the separation of two nearby geodesics as determined by the curvature of spacetime. The Riemann curvature tensor is given as,

$$R^{\mu}_{\nu\lambda\sigma} = \frac{\partial\Gamma^{\mu}_{\nu\sigma}}{\partial x^{\lambda}} - \frac{\partial\Gamma^{\mu}_{\nu\lambda}}{\partial x^{\sigma}} + \Gamma^{\mu}_{\lambda\epsilon}\Gamma^{\epsilon}_{\nu\sigma} - \Gamma^{\mu}_{\sigma\epsilon}\Gamma^{\epsilon}_{\nu\lambda} \tag{53}$$

The Riemann curvature is a tensor, because it provides the coordinate basis components of the equation of geodesic deviation (51). The Riemann curvature tensor has dimensions of $\frac{1}{\text{length}^2}$, and gives a local measure of curvature.

3.3.2 The Einstein Equation

The Einstein equations are ten non-linear partial derivative equations of the metric $g_{\mu\nu}(x)$ given the matter sources $T_{\mu\nu}(x)$, given subsequently. They are analogous to the Maxwell equations in electrodynamics, but their non-linearity makes them much more difficult to solve. The Einstein equations may be expressed by the form:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{54}$$

where $T_{\mu\nu}$ is the stress-energy tensor, and the Einstein curvature tensor is expressed by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R. \tag{55}$$

Here, $R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$ is called the Ricci curvature, which is found by summarising over the repeated indices of equation (53), and the component $g_{\mu\nu}R$ contains the Ricci curvature scalar $R = R^{\lambda}_{\lambda} = g^{\lambda\sigma}R_{\lambda\sigma}$ [5, 22.3]. The stress-energy tensor, is a symmetric tensor with components describing energy density, energy flux, momentum density and the stress tensor.

$$T^{\mu\nu} = \begin{array}{c} \\ t \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} t & x & y & z \end{array} \\ \left(\begin{array}{c|ccc} \text{energy density} & & \text{energy flux} & \\ \hline & & & \\ \text{mom. density} & & \text{stress tensor} & \\ & & & \end{array}\right) \end{array} \tag{56}$$

This stress-energy is the source of spacetime curvature in equation (54). The stress-energy of matter is not conserved in curved spacetime; it changes in response to the dynamic curvature. This means, that the stress-energy tensor is not conserved in the general case. However, it is possible to consider local conservation of energy-momentum, which is written as,

$$\nabla_\nu T^{\mu\nu} = 0, \tag{57}$$

where ∇_ν is the covariant derivative (42). The form of the Einstein curvature tensor (55) is determined by this law of local conservation of energy-momentum; it is only the Bianchi identity, which fulfils local conservation:

$$\nabla_\nu (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) = 0. \tag{58}$$

3.4 The Linearised Einstein Equation

There is no general solution to the Einstein equations, but it is possible to make a complete analysis of the Einstein equation for geometries of near-flat spacetime; with the assumption of linearised gravitational waves. This equation is called the linearised vacuum Einstein equation, and can be used to describe the geometry of a linearised gravitational wave, as in [5, 21.5].

The linearised Einstein equation is obtained by inserting the metric of near flat spacetime (46) into the Ricci curvature tensor for vacuum, $R_{\mu\nu} = 0$, and then expanding the term to first order in $h_{\mu\nu}(x)$ [6, 7.27]. The first term is the Ricci curvature for flat spacetime, and this term vanishes since all components of $n_{\mu\nu}(x)$ are constant. The second term is the first-order perturbation in the Ricci curvature $\delta R_{\mu\nu}$, which is linear in $h_{\mu\nu}(x)$. The linearised vacuum Einstein equation is thus,

$$\delta R_{\mu\nu} = \frac{\partial \delta \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} - \frac{\partial \delta \Gamma^\lambda_{\mu\lambda}}{\partial x^\nu} = 0, \tag{59}$$

expressed in Christoffel symbols. The wave operator $\Box$, referred to as the *d'Alembertian*, is defined:

$$\Box \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu = -\frac{\partial^2}{\partial t^2} + \boldsymbol{\nabla}^2, \tag{60}$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$. This operator can be used to express the linearised vacuum Einstein equation,

$$\boxed{\delta R_{\mu\nu} = \frac{1}{2} \left[-\Box h_{\mu\nu} + \partial_\mu V_\nu + \partial_\nu V_\mu \right] = 0} \tag{61}$$

Here, the vectors denoted V_μ and V_ν are particular combinations of perturbations of the geometric metric, and have the form,

$$V_\mu \equiv \partial_\lambda h^\lambda_\mu - \frac{1}{2} \partial_\mu h^\lambda_\lambda \tag{62}$$

It is noted that $h^\lambda_\mu = \eta^{\lambda\sigma} h_{\sigma\mu}$ [5, 21.5].

3.4.1 Gauge Transformation

Since choice of coordinates is completely arbitrary, it is evidently best to choose coordinates, in which the solution of the linearised Einstein equation is simplified, as it is done in [6, 3.2] and [5, 21.5]. The ten independent linearised equations (61), cannot determine the perturbation $h_{\mu\nu}(x)$ uniquely, since the value of these may be changed without changing the geometry. The assumptions made in the metric perturbation (46), does not determine coordinates. It is possible to make small changes to the coordinates, making small changes in the perturbation $h_{\mu\nu}(x)$, but leaving the Minkowski metric (47) unchanged.
Consider a change of coordinates:

$$x^{'\mu} = x^{\mu} + \xi^{\mu}(x) \tag{63}$$

where $\xi^{\mu}(x)$ are four functions of the same size as the small perturbations $h_{\mu\nu}(x)$.
The metric generally transforms according to equation (39), which results in a transformed metric, that has the same form as the original, but with new perturbations given by the gauge transformation,

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu}. \tag{64}$$

The four arbitrary functions $\xi^{\mu}(x)$ may be chosen such that $V^{'}_{\mu}(x) = 0$. This would simplify the linearised Einstein equation (61) to becomes the wave equation,

$$\Box h_{\mu\nu}(x) = 0 \tag{65}$$

together with the Lorentz gauge conditions,

$$V_{\mu}(x) \equiv \partial_{\nu}h^{\nu}_{\mu}(x) - \frac{1}{2}\partial_{\mu}h^{\nu}_{\nu}(x) = 0. \tag{66}$$

3.4.2 Solving the linearised Einstein Equation

Perturbations of the metric are determined by the wave equation (65) and the gauge conditions (66). The general solution for the wave equation, can according to [5, 21.5] be determined by Fourier transformations, and has the form,

$$f(x) = \int d^3k a(k) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{67}$$

Here the integral runs over all possible wave numbers $\vec{k}$, and $a(\vec{k})$ are arbitrary complex amplitudes. To get physical solutions, ones takes the real part of (67). All components of the perturbation satisfy the flat spacetime wave equation according to (65). For a wave with a defined wave vector $\vec{k}$, the perturbation can be written in the matrix form,

$$h_{\mu\nu}(x) = a_{\mu\nu} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{68}$$

where $a_{\mu\nu}(x)$ is a 4×4 matrix containing the amplitudes of the various components of the wave. The perturbations of the metric must also satisfy the gauge conditions (66). These conditions allow for further coordinate transformations, which can be used to simplify the amplitude matrix $a_{\mu\nu}$. If a gauge transformation of the form (64), is made with $\xi_{\mu}(x)$, and inserted into the Lorentz gauge condition (66), one finds,

$$\Box \xi_{\mu}(x) = 0. \tag{69}$$

These transformations can be used to make any four of the $h_{\mu\nu}$ vanish, and $a_{ti} = 0$ is chosen. These result in the components V_t and V_i of vector V_μ (62), become,

$$V_t = \frac{\partial h_t^t}{\partial t} = i\omega_k a_{tt} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = 0, \quad V_i = \frac{\partial h_i^j}{\partial x^j} = ik^j a_{ji} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = 0, \tag{70}$$

This supports that gravitational waves are transverse waves, since $a_{tt} = k^j a_{ij} = 0$. If the $z-$axis is set to be along the direction of the propagating wave, such that the wave vector is $\vec{k} = (0, 0, \omega)$ with frequency ω, the general solution of the linearised Einstein equation is,

$$h_{\mu\nu}(x) = \begin{array}{c} \\ t \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} t & x & y & z \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} e^{i\omega(z-t)} \tag{71}$$

This is called the *transverse traceless gauge*, and it is exactly the same plane wave described in (50), where the amplitudes a and b are proportional to $+$ and $\times$ polarisation, respectively.

There is a simple logarithm which allows any plane wave, satisfying equations (65) and (66), to transform to the transverse traceless gauge. The method is to set all non-transverse components to zero, and subtract the trace of all the diagonal elements. For a gravitational wave propagating along the $z-$axis,

$$h_{\mu\nu}^{TT}(x) = \begin{array}{c} \\ t \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} t & x & y & z \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(h_{xx} - h_{yy}) & h_{xy} & 0 \\ 0 & h_{xy} & -\frac{1}{2}(h_{yy} - h_{xx}) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}. \tag{72}$$

3.5 Emission of Gravitational Waves

To interpret any observations of gravitational waves, it is important to solve the Einstein equation for gravitational radiation produced by a given source. Hence, it seems useful to consider how gravitational waves are emitted. In this section, the assumptions of linearised gravity are still valid, and the source of weak gravitational radiation is assumed to be non-relativistic

3.5.1 Linearised Einstein Equation with a Source

To find the linearised Einstein equation with a source, the same procedure is used as for the linearised vacuum equation. Small perturbations of spacetime are assumed, such that $g_{\mu\nu}(x)$ may be expressed as in (46). The left hand side of the Einstein equation is evaluated to the first order in the metric perturbation. Rectangular coordinates are assumed, the speed of light set to $c = 1$, and all velocities in the source are assumed to be much smaller than the speed of light. As a result, the stress energy is dominated by the rest mass density μ,

$$T^{\mu\nu} = \mu u^\mu u^\nu \tag{73}$$

It is possible to impose gauge conditions on the perturbation $h_{\mu\nu}(x)$, and the four conditions of the Lorentz gauge may be written simply as:

$$\frac{\partial \bar{h}^{\mu\nu}}{\partial x^{\nu}} = 0 \tag{74}$$

where $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{\lambda}_{\lambda}$ is the *trace-reversed* amplitude of the perturbation. The linearised curvature scalar is defined by (61), but is simplified by the trace-reversed expression of the Lorentz gauge (74) to become,

$$\delta R_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu}. \tag{75}$$

The linearised Einstein equation for weak sources may then be written as,

$$\boxed{\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}.} \tag{76}$$

3.5.2 The General Solution of Linearised Gravity

Each component of (76) obeys a flat wave equation with a source on the form:

$$-\frac{\partial^2 f(x)}{\partial t^2} + \vec{\nabla}^2 f(x) = j(x), \tag{77}$$

where $j(x)$ is the source of the wave equation for $f(x)$. The general solution for the wave equation is given by:

$$f(t,\vec{x}) = -\frac{1}{4\pi}\int d^3x' \frac{[j(t',\vec{x}')]_{ret}}{|\vec{x}-\vec{x}'|}, \tag{78}$$

where $[\ \cdot\ ]_{ret}$ means, that the expression must be evaluated at the retarded time $t' = t_{ret} \equiv t - |\vec{x}-\vec{x}'|$. The general solution (78) is for a wave equation with a source and outgoing wave boundary conditions; that is, a wave propagating outwards from the source. The retarded wave is emitted *after* the event that is its source [5, 23.1].

The general solution (78) can be immediately applied to to the case of linearised gravity. By raising the indices on the linearised Einstein equation with a source (61), as set of 10 wave equations $\bar{h}^{\mu\nu}$ with $-16\pi T^{\mu\nu}$ as a source, and hence the general solution of the linearised Einstein equation given by:

$$\bar{h}^{\mu\nu}(t,\vec{x}) = 4\int d^3x' \frac{[T^{\mu\nu}(t',\vec{x}')]_{ret}}{|\vec{x}-\vec{x}'|}, \tag{79}$$

When the source varies harmonically in time with the frequency ω, and the gravitational wave has wavelength $\lambda = \frac{2\pi}{\omega}$, and R_{source} are the characteristic dimensions of the source, the long wavelength approximation is:

$$\lambda \gg R_{source} \tag{80}$$

From (79), the asymptotic gravitational wave amplitude can be found for $r \to \infty$. The amplitude (79) in the long wavelength approximation (80) and $r \gg R_{source}$, can be simplified, because $|\vec{x}-\vec{x}'|$ can be replaced by r in the denominator.:

$$\bar{h}^{\mu\nu}(t,\vec{x}) \xrightarrow[r\to\infty]{} \frac{4}{r}\int d^3x' T^{\mu\nu}(t-r,\vec{x}'). \tag{81}$$

Over a limited angle, the wave described by the amplitude (81) is approximately a plane wave. To get the wave amplitude, it is only necessary to consider the spatial components of source of the plane wave. For a plane gravitational wave, the spatial components are expressed by:

$$\int d^3x T^{ij}(x) = \frac{1}{2}\frac{d^2}{dt^2}\int d^3x x^i x^j T^{tt}(x) \tag{82}$$

where the index ij represents the spatial components of the tensor, and the index tt is the time component. Assuming that the stress energy tensor, takes the same form as (73), the integral will be dominated by the rest mass energy $\mu(x)$. This makes it possible to simplify expression for the amplitude, such that it becomes,

$$\boxed{\bar{h}^{ij}(t,\vec{x}) = \xrightarrow[r\to\infty]{} \frac{2}{r}\ddot{I}^{ij}(t-r),} \tag{83}$$

where $\ddot{I}^{ij}$ is the second mass moment differentiated twice with respect to time t. The second mass moment is defined to be,

$$I^{ij}(t) \equiv \int d^3x\mu(t,\vec{x})x^i x^j. \tag{84}$$

3.5.3 The Quadrupole Formula

Gravitational waves carry energy away from a radiating system. The total rate of energy loss, for gravitational radiation in the weak-field, long wavelength approximation, is given by an expression called the quadrupole formula. The formula is derived by calculating the energy flux in different directions and integrating over a solid angle, but it's form can be anticipated in a few steps, as it is done by [5, 23.6]. The energy flux of a linearly polarised plane gravitational wave [5, 16.5] with frequency ω and amplitude a is given by:

$$f_{GW} = \frac{\omega^2 a^2}{32\pi}. \tag{85}$$

In the weak source, long wavelength approximation for a source far away, the amplitude is expressed in terms of the second derivative of the mass moment $\ddot{I}^{ij}$, as it is done in equation (83). The energy flux (85) contains a factor ω^2, which makes one more time derivative for each of the two factors of the second mass moment; the total radiated power L_{GW} therefore dependent on the third derivative of the mass moment $\dddot{I}^{ij}$. The total radiated power behaves as a scalar under spatial rotation, and hence it must be a quadratic scalar combination of $\ddot{I}^{ij}$.
There are two possible combinations: $\dddot{I}^{ij}\dddot{I}_{ij}$ or $(\dddot{I}^k_k)^2$. The quadrupole moment tensor is,

$$\tilde{I}^{ij} \equiv I^{ij} - \frac{1}{3}\delta^{ij}I^k_k, \tag{86}$$

and vanishes in a spherically symmetric system, where $I^{ij} \propto \delta^{ij}$. Hence the total radiated power must be proportional to $\dddot{I}^{ij}\dddot{I}_{ij}$. The total radiated power, also called the luminosity L_{GW}, has the form:

$$L_{GW} = \frac{1}{5}\langle \dddot{\tilde{I}}_{ij}\dddot{\tilde{I}}^{ij}\rangle, \tag{87}$$

in geometrised units, where $\langle\cdot\rangle$ denotes the time average, and the proportionality constant is $\frac{1}{5}$.

3.6 Dipole Versus Quadrupole Radiation

The gravitoelectric analogy is a good description of gravity, because the basic field equations fields of both gravity (54) and electromagnetism (24) can be reduced to the wave equation with a source in an appropriate gauge. However, there are some crucial differences between electromagnetic fields and gravitational fields; not at least the fact that an electromagnetic field is a vector field, whereas the gravitational field is a tensor field. Both types of fields can be expanded for $\frac{1}{r}$ in

the long wavelength approximation far from the source, and the coefficients of these multipole expansions are proportional to the time derivatives of their multipole moments [5, 23.4]. One significant difference between electromagnetism and gravity is, that electric charges be either positive or negative, while mass only has one sign. This means, that the gravitoelectromagnetic analogy is suitable for electromagnetism with oppositely charged particles only. This fact has consequences for the type of radiation emitted.
In electromagnetism, the dominant radiation for a moving charge is dipole radiation. For an electric charge density $\rho_e(\vec{r})$, the monopole moment is the total charge Q. The charge must be conserved, and hence there can be no electromagnetic monopole radiation. The static electric dipole moment:

$$\vec{p} = \int \rho_e(\vec{r})\vec{r}d^3r, \tag{88}$$

is however, not conserved. The lowest possible moment will always be the dominating one, unless it is eliminated by some symmetry. The electromagnetic dipole radiation is hereby the dominating radiation factor. In the section on the *Gravitoelectric Analogy* 2.3, a gravitational wave is assumed to be emitted by dipole radiation. However, as already established in section 3.5.3, the luminosity of the radiated energy of a gravitational wave (87) is dependent on the *quadrupole* moment tensor (86). This is because conservation of momentum eliminates dipole radiation, and quadrupole radiation becomes the dominating factor instead. The gravitational monopole moment of the mass energy $\mu(\vec{r})$, is the total mass-energy, which must be conserved. The gravitational static dipole moment has the same form as (88), and is given by:

$$\mu = \int \mu(\boldsymbol{r})\boldsymbol{r}d^3r. \tag{89}$$

The dipole moment (89) is the centre of the mass energy of the system, which is constant in the centre of mass reference frame. The chosen coordinate system can always be defined such, that the origin is at the centre of mass. Here, the dipole moment is zero, and no dipole radiation is emitted [14, 4.7]. The next multipole is the quadrupole moment, as in equation (84):

$$I^{ij}(\boldsymbol{r}) = \int \mu(\boldsymbol{r})r^i r^j d^3r. \tag{90}$$

The quadrupole moment is not conserved, and hence the simplest form of non-zero luminosity radiation is quadrupole radiation. This indicates which types of objects might emit gravitational radiation. It can be shown that the quadrupole of a spherically symmetric mass is zero, and thus spherically symmetric metric perturbations $h_{\mu\nu}(x)$ do not produce gravitational radiation up to the quadrupole moment [13]. No matter how violent an explosion or collapse, gravitational waves will not be emitted, if spherical symmetry is conserved. Possible sources of gravitational radiation includes asymmetrical collapses, binary systems and rotating stars with non-axisymmetric shapes.

4 Detection of Gravitational Waves

4.1 Indirect Measurements

The formula (87) can be used to calculate the total power radiated for a binary system, in units $c = 1$, $G = 1$. Consider a binary system of two equal masses M with a distance R to their shared

centre of mass and an orbital frequency Ω, as seen in figure (7). The geometry of this system may be written as:

$$x(t) = R\cos\Omega t, \quad y(t) = R\sin\Omega t, \;\; z(t) = 0. \tag{91}$$

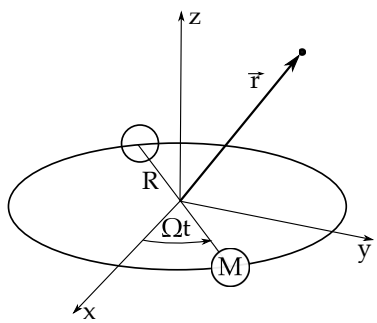

Figure 7: Two bodies of equal mass M orbit at a radius r with an orbital frequency Ω around a common centre of mass. The gravitational radiation from this binary system is detected far away in any direction $\vec{r}$. Adapted from [5, Figure 23.3]

Using equation (84), the components of the second mass moment of this system can be determined to be:

$$I^{xx} = 2MR^2\cos\Omega t^2, \;\; I^{xy} = 2MR^2\sin\Omega t\cos\Omega t, \quad I^{yy} = 2MR^2\sin\Omega t^2. \tag{92}$$

The trace for this system $I^k_k = 2MR^2$, is independent of time, which means that $\dddot{\tilde{I}}^{ij} = \dddot{I}^{ij}$. Summing over the squares of the components of the three times differentiated quadrupole moment tensor (86), the total radiated power of the binary system can be determined to be:

$$L_{GW,BS} = \frac{128}{5}M^2R^4\Omega^6. \tag{93}$$

The radiated power (93) may be written such that it depends on the orbital period $P = \frac{2\pi}{\Omega}$, using Kepler's law $R = \left(\frac{M}{4\Omega^2}\right)^{1/3} = \left(\frac{MP^2}{16\pi^2}\right)^{\frac{1}{3}}$, it becomes:

$$L_{GW,BS} = \frac{128}{5}4^{\frac{1}{3}}\left(\frac{\pi M}{P}\right)^{\frac{10}{3}} \tag{94}$$

This radiated power determines the energy radiated from a binary system, consisting of two objects with equal mass in circular orbits.

4.1.1 Hulse-Taylor Binary

In 1973, professor Joseph Taylor and Russell Hulse were the first to detect a binary pulsar system sketched in figure (9), which they named *PSR 1913+16*. In accordance with section 3.6, a binary pulsar system emits quadrupole gravitational radiation, and thus the 1973 discovery provided the first opportunity to indirectly detect a gravitational wave.

A pulsar is a rapidly spinning neutron star, which is created in a supernova, during the late life of a massive star. The massive star, having used up all its fuel, collapses in on itself and causes an explosion, which throws off the outer layers of the star. The conservation of angular momentum and magnetic field, causes the dense core to rotate along its intense magnetic field.

The combination of a strong magnetic field and pulsar rotation gives rise to the emission of narrow beams of radiation near the poles, as demonstrated in figure (8), resulting in a lighthouse effect [15]. If a pole spins to face the earth during its rotation, it is possible to detect the pulse of energy emitted. A single pulsar spins with an extremely stable period, and a pulsar can therefore be used as a precise clock. Hulse used the largest available radio telescope, the 305 meter Arecibo telescope, which has a high time resolution; optimal for measuring low-frequency pulsar signals. As the pulsars orbit each other in a binary system, the two stars slowly spiral towards each other, as the orbital energy is converted to gravitational radiation. This loss of orbital energy and angular momentum will cause the orbital period to decrease over time. By measuring the cumulative shift of the time between periastron position over time, the loss of orbital energy is indirectly measured. In the Newtonian approximation appropriate for the non-relativistic binary system, the energy is:

$$E_{Newton} = 2\left(\frac{1}{2}MV^2\right) - \frac{M^2}{2R} = -\frac{1}{4}M\left(\frac{4\pi M}{P}\right)^{\frac{2}{3}} \tag{95}$$

where V is the orbital speed and M is the mass of one of the pulsars. Differentiating the Newtonian energy (95) with respect to time t, and setting this equal to the radiated power (94), an expression for the rate of change in the period is found to be:

$$\frac{dE_{Newton}}{dt} = L_{GW,BS} \Rightarrow \frac{dP}{dt} = -\frac{96}{5}\pi 4^{\frac{1}{3}}\left(\frac{2\pi M}{P}\right)^{\frac{5}{3}}. \tag{96}$$

Although equation (96) is valid for a circular orbit, and this is not the case for the binary system PSR 1913+16, the decrease in orbital period of the system can be estimated from this equation as of order 1×10^{-6} s per year. In their published paper on *Measurements of general relativistic effects on binary pulsar PSR 1913+16* from 1979 [16], they found the rate of change in orbital period to be $(-3.27 \pm 0.6) \times 10^{-12}$ s s^{-1}. Measurements made over several decades, even after their first published paper, yielded the results seen in figure (10).

By 1979, Hulse and Taylor made approximately 1000 measurements of the rate of change in orbital period in the course of 4.1 years. Their data analysis demonstrated, that the loss of orbital energy was in complete agreement with the predictions made by equation (96). They concluded that gravitational waves do exist, and carry energy away from the binary system [16]. The decrease in orbital period in the binary pulsar system was the first experimental evidence for the existence of gravitational waves, and in 1993 Hulse and Taylor were jointly awarded the Nobel Prize in Physics "*for the discovery of a new type of pulsar, a discovery that has opened up new possibilities for the study of gravitation*" [17].

4.2 Direct Measurements

Although the Hulse-Taylor binary system allowed for the first confirmation of the existence of gravitational waves, the quest for direct measurements of the effect of gravitational waves continued. Early attempts to measure gravitational radiation were based on detecting the stretching and compressing force, a gravitational wave would apply to a great cylindrical resonance bar [18]. Highly sensitive accelerometers would be mounted on each end of the bar, and were supposed to detect the vibrations caused by the propagating gravitational wave. However, to this date it has

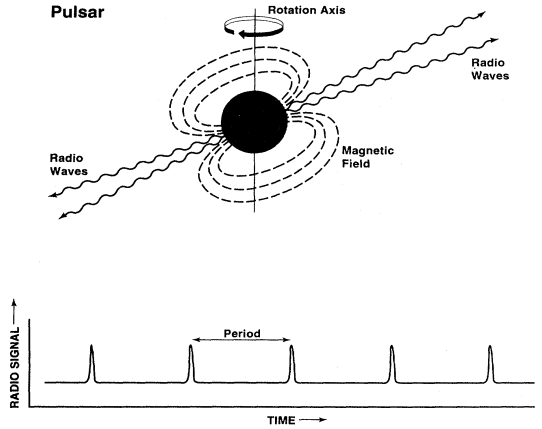

Figure 8: A conceptual sketch of a rotating pulsar, in the form of a rapidly rotating neutron star. Narrow beams of radio wave radiation are emitted from the magnetic poles. The diagram at the bottom, demonstrates the pulses of radio signals detected by a telescope on Earth [15, Figure 1].

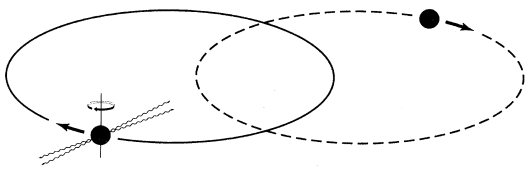

Figure 9: Conceptual drawing of a pulsar binary system of two pulsars of approximately equal mass [15, Figure 11]. The Periastron is the point on the orbit closest to the centre of mass.

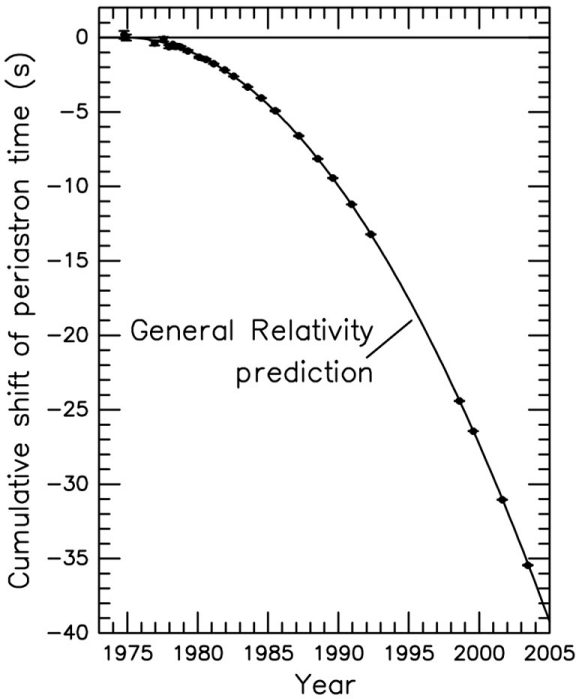

Figure 10: Detected decrease in orbital periode measured over decades. The data points are the decrease in orbital time for each year, while the solid line represents the loss in orbital energy predicted by general relativity. The agreement between data points and the theoretical values is better than one third of a percent [5, Figure 23.3].

not been possible to detect the effects of a gravitational wave using this method. Using a laser interferometer with freely falling test masses would prove to be a much more successful method for detecting the strain of spacetime. Interferometers have some significant advantages compared to the previously used resonant bars: they can register the actual change in separation between the test masses; and to greater effect, due to the large separation between the test masses, and hence the greater strain of the gravitational wave [19]. The laser interferometer gravitational-wave observatory *LIGO* successfully detected the strain caused by a gravitational wave in September 2015 [26].

4.2.1 LIGO

LIGO is the world's largest gravitational wave observatory, consisting of two identical twin detectors placed 3002 km apart at Hanford and Livingston. The detectors can be described as a sophisticated version of the Michelson interferometer. Each detector consists of two straight 4 km long steel vacuum tubes with a 1.2 m diameter, placed perpendicular and leveled to each other [20]. As in a simple Michelson set up seen in figure (11), a mirror is placed at the far end of each arm, and a beamsplitter is placed where the arms meet. The distance the laser beams have to travel along the arms is such, that total destructive interference occurs [21]. Michelson interferometry is

particularly well suited for detecting gravitational waves, because of the transverse nature of the tidal force produced by the waves [22].

A gravitational wave, propagating through the observatory, changes the proper distance of the steel arms and thus changes the separation between the beam splitter and the mirrors. A laser beam propagating in the arm experiences a change in wavelength proportional to the amplitude of the gravitational wave strain. The change in wavelength is analogous to the redshift of light caused by the expansion of the Universe: but in contrast to the universal expansion, the change in separation is polarised and happens instantly. This means, that the electromagnetic wave crests present in the steel arm, when the gravitational wave hits, experiences a change in separation. This change in crest separation means, that these crests will arrive at the beam splitter either later or sooner than expected, depending on whether the separation becomes larger or smaller. The laser beams will no longer be completely out off phase when they arrive at the beamsplitter, and a phase shift is detected as seen in figure (12). This change in separation however, only affects the laser beams in the arm at the time, and any light emitted after this event will have the original wavelength. By measuring the time it takes before light arrives, which has not experienced a change in wavelength, it is possible to determine by which factor the length of the arm has changed [19].

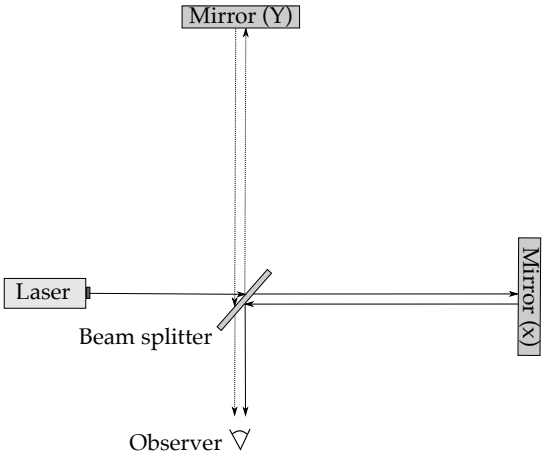

Figure 11: Simplified conceptual diagram of LIGO. A laser beam is emitted and split into two perpendicular arms. The beams are reflected by the freely falling mirrors, and travel through the beam splitter before they reach the observer. The length of the arms are such, that destructive interference occurs. Adapted from [22, Figure 6]

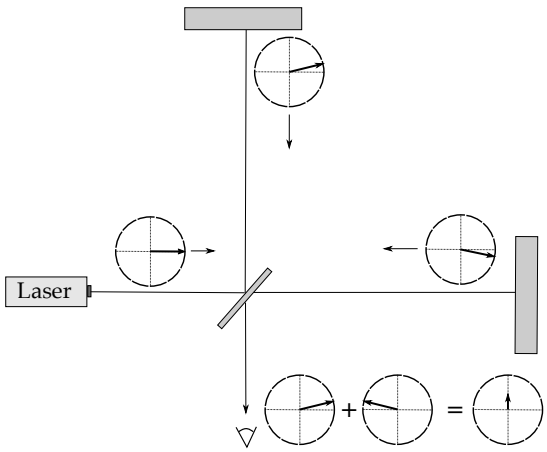

Figure 12: Conceptual drawing showing change in the laser phase. One arm is lengthened while the other is shortened by the same factor, resulting in equal but opposite phase shifts. The beam splitter introduces an additional $180°$ phase shift, resulting in an output beam with a $90°$ and smaller amplitude than the emitted signal. Adapted from [22, Figure 5]

In contrast to other astronomical observatories, LIGO does not observe electromagnetic radiation from space, but rather uses laser light to detect strain caused by a gravitational wave. Because of this, the vacuum steel arms are isolated from the outside world by huge concrete walls, in order to prevent noise from entering the observatory. However, the detectors are so sensitive, that they may pick up some local sources such as earthquakes or acoustic noise, even through the concrete walls. As a result, it is necessary to make measurements using twin detectors. When the two detectors are separated by a great distance, they will not detect the same local vibrations. A gravitational wave will reach the two detectors with almost no time delay, and the detectors should measure the same strain. In this way, it is possible to distinguish measurements of gravitational waves and

vibrations from local forces [23].

4.2.2 Detecting Gravitational Wave Strain

As already mentioned, the laser interferometer LIGO detects a phase shift of the laser light, caused by the change in proper length of the steel arms. In other words, the phase shift is the physical observable that indicates the presence of a gravitational wave. This statement holds true if the mirrors are free to move along the axes of the arms, and is expressed in the local Lorentz gauge, that is the reference frame of the observatory. It is however, easier to work in the transverse traceless gauge (71), since a gravitational wave takes a simple form in this gauge, and the coordinates of the mirrors do not change, even when the separation between them changes. In the transverse traceless gauge, the amplitude of the gravitational wave may be written as the perturbation element $h_{\mu\nu}$ from equation (46),:

$$h_{\mu\nu} = \begin{array}{c} \\ t \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} t & x & y & z \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} e^{-i(\Omega t + k_\Omega z)} \tag{97}$$

where Ω is the angular frequency of the gravitational wave, $k_\Omega = \Omega/c$ is the wave number of the gravitational wave, and h is components of the amplitude. The geometry of the spacetime is given by the infinitesimal interval,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{98}$$

It is possible to calculate the phase shift $\Delta\phi$ caused by a gravitational wave normally incident on the observatory [18]. It is assumed that the wavelength of the gravitational wave, is much greater than the length L, of the interferometer's arms, $\Omega L/c \gg 1$, and calculations are to first order in the small dimensionless amplitudes of the wave. Lasers travel along lightlike geodesics, which means that $ds^2 = 0$, and if the arms are defined to be the x- and y-axis with the origin at the beamsplitter, equation (98) becomes,

$$g_{00} c^2 dt + g_{11} dx^2 + g_{22} dy^2 + 2 g_{12} dx dy = 0. \tag{99}$$

For propagation of light along the x-axis only, that is, the y-components are set to zero, the following relation is obtained:

$$c \left| \frac{dt}{dx} \right| = \left| \frac{g_{11}}{g_{00}} \right|^{\frac{1}{2}} = [1 + h_{11}(t)]^{\frac{1}{2}} \approx 1 + \frac{1}{2} h_+ e^{-i\Omega t}, \tag{100}$$

where the last approximation follows from the binomial theorem. The time interval for the light to travel from the beam splitter along the x-axis, at time t_0 for it to return at t_r is given by:

$$\Delta t_x = t_r - t_0 = \int_0^L \left| \frac{dt}{dx} \right| dx - \int_L^0 \left| \frac{dt}{dx} \right| dx, \tag{101}$$

where the minus sign accounts for $dt/dx = -|dt/dx|$ on the return trip from the mirror. When doing the integration, it is possible to replace t with x/c, since any corrections will be of the

second order of h. Furthermore, if the one way travel time is defined as $T = L/c$ before the gravitational wave hits the observatory, the integration evaluated at t_r gives:

$$\Delta_x t(t_r) = \frac{2L}{c} + h_{11}(t_r)\frac{L}{c}\frac{\sin \Omega T}{\Omega T}e^{i\Omega T}. \tag{102}$$

If the same integral is made for the y-axis, the result is the same as equation (102) where $h_{11}(t)$ is replaced by $-h_{22}(t)$. Hence, the difference in travel time between the two arms is:

$$\Delta t(t) = \Delta t_x(t) - \Delta t_y(t) = 2[h_+ e^{i\Omega t}]\frac{L}{c}\frac{\sin \Omega T}{\Omega T}e^{i\Omega T}. \tag{103}$$

If the angular frequency of the laser is ω_0, the phase shift corresponding to a delay of time Δt, is given by $\Delta\phi = \omega_0 \Delta t$. The observable phase shift is therefore,

$$\Delta\phi = 2\omega_0[h_+ e^{i\Omega t}]\frac{L}{c}\frac{\sin \Omega T}{\Omega T}e^{i\Omega T}. \tag{104}$$

The phase shift (104) varies with the factor $\sin \Omega T/\Omega T$, and this is why the laser light must have a frequency much higher than the gravitational wave [18].

4.3 Chirping Waveform

Twice, since the advanced LIGO observatory initiated data collection, the strain of spacetime caused by a gravitational wave has been successfully detected. The observation GW150914 took place on the 14th of September 2015 [24] and the observation GW151226 was made on the 26th of December 2015 [25]. Coincidentally both signals were emitted from binary black holes orbiting each other approximately 1.3 billion light years away. Due to the weak interaction between the gravitational field and matter, LIGO only detected radiation emitted during the final orbits of two black holes just before and at their coalescence. The first detection measured a one-ten-thousandth of the diameter of a proton change in separation between the bam splitter and the mirror, that is a change to the order 10^{-19}m [26]. The signal of the first detection at Handford is plotted as the blue line in figure (13), and here the signal frequency increases from 35Hz to 250Hz in 0.4s, and the peak gravitational wave strain is found to be $\approx 1.0 \times 10^{-21}$ [24]. The strain data from the Hanford detection of GW150914 is found on the *LIGO Open Science Center* webpage [27].

The orbiting black holes spiral inwards towards each other while converting orbital energy to gravitational radiation; in the same manner as the binary pulsar system discovered by Hulse and Taylor. The frequency and strength of the quadrupole radiation increases with time, as the black holes orbit each other extremely fast before finally merging; resulting in a gravitational wave signal with a chirping waveform. Using time series analysis techniques, such as Fourier transformations and matched filtering, LIGO is able to extract astrophysical parameters from the chirping binary signal detected. These parameters of the binary black hole system can be used to simulate the chirping signal of the detected strain [28].
For a circular binary system, where the components are considered to be point particles, the time dependent orbital period decreases with time according to:

$$P_{orb}(t) = (P_0^{8/3} - \frac{8}{3}kt)^{3/8}, \tag{105}$$

where P_0 it the orbital period at time $t = 0$. The evolution constant k is given by:

$$k \equiv \frac{96}{5}(2\pi)^{8/3}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3}. \tag{106}$$

Here $\mathcal{M} \equiv (M_1 M_2)^{3/5}(M_1 + M_2)^{-1/5}$ is the chirp mass, constructed from the individual masses M_1 and M_2 of the two orbiting bodies. The time dependent orbital period, equation (105), can be determined by integrating (96) and realising that $\mathcal{M} = M/2^{1/5}$ when $M = M_1 \approx M_2$. Units of $c \neq 1$, $G \neq 1$ are obtained by letting $M \to GM/c^2$ and $P \to cP$.
To the leading order in the gravitational wave production, it is only possible to determine the chirp mass $\mathcal{M}$ of a binary system, and not the individual bodies. Because of this, it is not possible to distinguish between binary systems with the same chirp mass. It is important to note that equation (105) gives the period for the orbiting binary system, not the emitted gravitational waves. The time dependent period of the emitted gravitational waves can be expressed in terms of the orbital period:

$$P_{GW}(t) = \frac{1}{2}P_{orb}(t). \tag{107}$$

Here, the factor one half is due to the quadrupole order of gravitational radiation; since the quadrupole moment is invariant under a 180° rotation, the gravitational wave period yields a factor of two per complete orbit. The gravitational wave strain $h(t)$ produced as the bodies spiral inwards, can be expressed as:

$$h(t) = \mathcal{A}(t)\cos\Phi(t), \tag{108}$$

where $\mathcal{A}(t)$ is the time dependent amplitude and $\Phi(t)$ is the gravitational wave phase. The gravitational wave phase is analogous to the phase for other wave phenomenon, and it is related to the period of the gravitational wave P_{GW} by the integral:

$$\Phi(t) = \Phi_0 + 2\pi\int_0^t \frac{dt'}{P_{GW}(t')}. \tag{109}$$

Here Φ_0 is the initial phase. The time dependent amplitude can be expressed in terms of the astrophysical parameters of the binary system:

$$\mathcal{A}(t) = \frac{2(G\mathcal{M})^{5/3}}{c^4 r}\left(\frac{\pi}{P_{GW}(t)}\right)^{2/3}, \tag{110}$$

where r is the luminosity distance to the binary system.
Using the parameters for the detection GW150914 as listed in table 1 in the discovery paper [24], a chirping signal as seen in figure (15) is simulated in MatLab. The black holes in this binary system have the masses $M_1 = 36^{+5}_{-4}M_\odot$ and $M_2 = 29^{+4}_{-4}M_\odot$ and are at a luminosity distance $r = 401^{+160}_{-180}$Mpc. The simulated signal (15) shows the emitted gravitational radiation leading to the coalescence of the two black holes. It is noted that LIGO only detected the last 0.5s of the signal. The part of the signal detected by LIGO is the characteristic part of the chirp signal, and where the changes in amplitude and frequency is most prominent. The gravitational wave signal becomes more monochromatic further away from the coalescence. Here the binary evolution happens so slowly, that the frequency derivative term is undetectable, and the amplitude is a fraction of the peak gravitational wave strain. In figure (13), the whole detected signal is plotted, and here it is noticeable that the simulated signal is not in agreement with the detected data

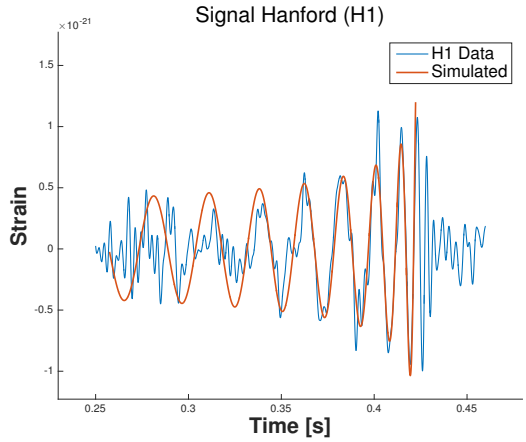

Figure 13: The detected and simulated signals during the 0.5 seconds leading to and including the coalescence.

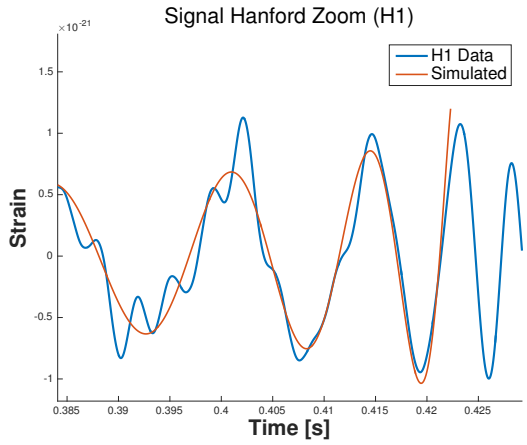

Figure 14: The detected and simulated signals at the coalescence.

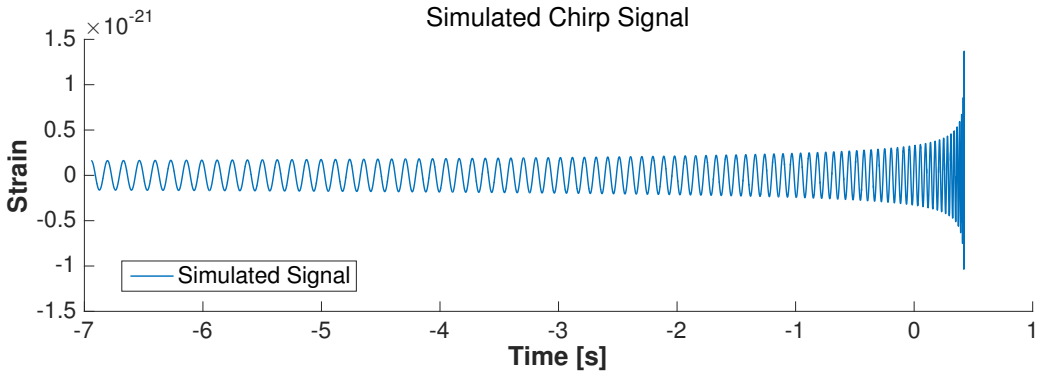

Figure 15: The simulated signal over a larger period of time before the coalescence of the two black holes. The signal evolves from a near monochromatic signal to a prominent chirp signal in the final seconds.

throughout the entire signal. In the time interval 0.25s – 0.35s the gravitational wave signal is difficult to distinguish from the noise, due to the small strain amplitude $\mathcal{A}(t)$. In figure (14), the simulated signal is in compliance with the detected signal at the coalescence, where the change in frequency is almost identical. There is however, a deviation between the amplitudes of the signals, which might be caused by noise in the detected signal. Overall, the simulated signal is in good agreement with the detected gravitational wave strain. The simulated wave is able to model how the binary system evolves through time, when the gravitational wave strain is too small for LIGO to detect.

4.4 Future Detections

The first run of the advanced LIGO took place from September 2015 to January 2016, following which scientists evaluated it's performance and improved the equipment [29]. On the 30th of November 2016 LIGO resumed the search for gravitational waves with improved sensitivity [30]. So far, gravitational waves emitted by merging black holes have been detected, but in theory there are many other possible sources; such as the explosion of a supernova, the rotation of a non-spherical neutron star, or even radiation from the Big Bang. A cosmic object, yet to be detected, is an

intermediate mass black hole (IMBH). Until recently, it has been believed that black holes naturally form in two mass ranges: stellar black holes in the range $M \sim 3M_\odot - 20M_\odot$, and supermassive black holes with a mass $M \sim 10^6 M_\odot - 10^{10} M_\odot$ [31]. However, recent evidence suggests, that there might exist black holes with an intermediate mass in the range $M \sim 10^2 M_\odot - 10^4 M_\odot$ [32]. It is not possible to directly detect an IMBH by use of electromagnetic radiation, but LIGO might detect a gravitational waves chirp signal from an IMBH binary system, or a binary system with an IMBH and a stellar black hole. Figure 16 and figure 17 are created in MatLab, and illustrate the expected strain caused by gravitational waves emitted from IMBH binary systems.

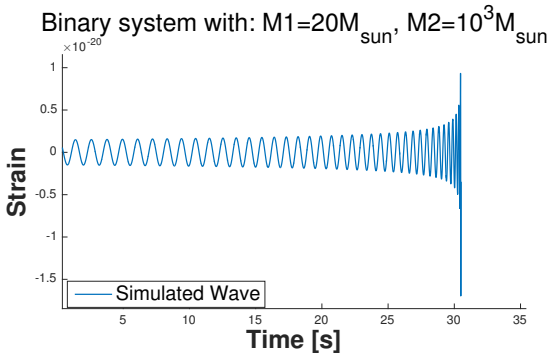

Figure 16: Simulated gravitational wave chirp signal for a stellar black hole and IMBH binary system, during the last 30 seconds before coalescence.

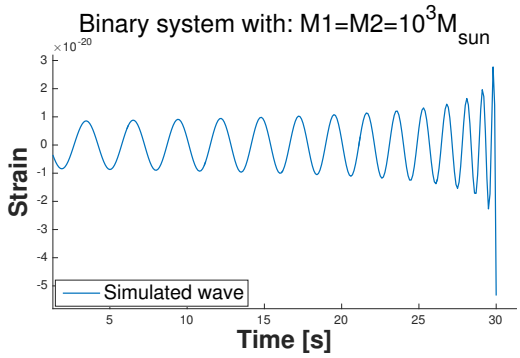

Figure 17: Simulated gravitational wave chirp signal for two IMBHs with equal mass in a binary system, during the last 30 seconds before coalescence.

At this point in time, LIGO is sensitive to gravitational radiation with a frequency in the range $10Hz - 1000Hz$ [33]. The simulated signals for IMBH binaries seen in figure 16 and 17 have frequencies below the detector's sensitive band. Because of this, LIGO is not able to detect these signals, even though the expected strain has an amplitude 10 times greater than the detect signal GW150914.

Future detectors, such as the laser interferometer space antenna *LISA* and the proposed ground-based Einstein telescope, promises to achieve the sensitivity required to detect gravitational radiation from IMBH and supermassive black holes [34]. Apart from confirming the existence of some gravitational radiating sources, perhaps it will also be possible to learn more about the properties of gravitational waves, and determine whether or not the theory of general relativity is the correct theory of gravity. Unexpected discoveries are often made in science, when something is observed with a new perspective; so it is not unreasonable to anticipate the discovery of new phenomenons. Looking towards the future, it is not known what will be learned from gravitational radiation in the years to come. However, it is certain, that the detection of gravitational waves has opened up for a whole new field of enquiry, and with it comes exciting possibilities.

Bibliography

[1] **Caltech and MIT;** ***Gravitational Waves***, *Web. Jan. 2017,* https://www.ligo.caltech.edu/page/gravitational-waves.

[2] **A. Einstein,** ***Approximative Integration of the Field Equations of Gravitation***, *"Volume 6: The Berlin Years: Writings, 1914-1917", Edited by Martin J. Klein, A. J. Kox, and Robert Schulman, Princeton University Press, 1997.*

[3] **U. I. Uggerhøj,** ***Speciel Relativitetsteori***, *Aarhus Universitetsforlag, 2016.*

[4] **O. Heaviside,** ***Electromagnetic Theory***, *"The Electrician" Printing and Publishing Co. Limited, 1893, Salisbury, Fleet Street, London.*

[5] **J. B. Hartle,** ***Gravity: an introduction to Einstein's General Relativity*** *First Edition, Pearson Education Limited, 2003.*

[6] **H. C. Ohanian,** ***Gravitation and Spacetime***, *Rensselaer Polytechnic Institute, W. W. Norton and Company, Inc. 1976.*

[7] **L. Sartori,** ***Understanding Relativity: A Simplified Approach to Einstein's Theories***, *University of California Press, Berkley and Los Angeles California, 1996.*

[8] **D. J. Griffiths,** ***Introduction to Electrodynamics***, *Fourth Edition, Pearson Education Limited, 2014.*

[9] **D. Bedford og P. Krumm,** ***On relativistic gravitation***, *Am. J. Phys. 53, 889 (1985); doi:* http://dx.doi.org/10.1119/1.14358.

[10] **K. T. McDonald,** ***Answer to Question #49. Why c for gravitational waves?***, *Am. J. Phys. 65, 591 (1997); doi:* http://dx.doi.org/10.1119/1.18666.

[11] **R. T. Hammond,** ***Answer to Question #49. Why c for gravitational waves?***, *Am. J. Phys. 65, 591 (1997); doi:* http://dx.doi.org/10.1119/1.18666.

[12] **Young and Freedman, "University Physics"**, *Thirteenth Edition, Pearson Education Limited, 2014.*

[13] **M. Hendry,** ***An introduction to General Relativity, Gravitational Waves and Detection Principles***, *"SUPA Gravitational Wave Detection", University of Glasgow, Autumn 2012,* http://www.astro.gla.ac.uk/~martin/teaching/gravitational-wave-detection/supa_gwd_mah.pdf.

[14] **D. Green,** ***Lectures in Particle Physics***, *"World Scientific Lecture Notes in Physics - vol. 55", World Scientific Publishing Co. Pte. Ltd., 1994.*

[15] **R. A. Hulse,** ***The discovery of the binary pulsar***, *Princeton University, Plasma Physics Laboratory, Princeton, New Jersey 08543; doi:* https://doi.org/10.1103/RevModPhys.66.699.

[16] **J.H. Taylor, *Measurements of General Relativistic Effects in the Binary Pulsar PSR1913+16***, *Nature, 277, 437. 1979, doi:* http://dx.doi.org/10.1038/277437a0.

[17] **Nobelprize.org, *"The Nobel Prize in Physics 1993"*,** *Nobel Media AB 2014, Web. Dec 2016,* http://www.nobelprize.org/nobel_prizes/physics/laureates/1993/.

[18] **A. Melissinos, *The response of laser interferometers to a gravitational wave***, *Am. J. Phys. 78, (2010), Web. Dec 2016, doi:* http://dx.doi.org/10.1119/1.3443566.

[19] **P. Saulson, *If light waves are stretched by gravitational waves, how can we use light as a ruler to detect gravitational waves?***, *Am. J. Phys. 65, 501 (1997); doi:* http://dx.doi.org/10.1119/1.18578.

[20] **Caltech and MIT, *LIGO's Interferometer***, *Web. Dec 2016,* https://www.ligo.caltech.edu/page/ligos-ifo.

[21] **LIGO: Laser Interferometer Gravitational-Wave Observatory, *Parallels with light***, *13 October 2016,* https://www.ligo.caltech.edu/page/what-is-interferometer.

[22] **E. D. Black, *An introduction to signal extraction in interferometric gravitational wave detectors***, *American Association of Physics Teachers, Am. J. Phys., 71, (2003), doi:* http://dx.doi.org/10.1119/1.1531578.

[23] **Caltech and MIT, *LIGO's Dual Detectors***, *Web. Jan 2017,* https://www.ligo.caltech.edu/page/ligo-detectors.

[24] **B. P. Abbott, LIGO Scientific Collaboration and Virgo Collaboration, *Observation of Gravitational Waves from a Binary Black Hole Merger***, *American Physical Society, Phys. Rev. Lett. 116, 061102, 11th February 2016, doi:* https://doi.org/10.1103/PhysRevLett.116.061102.

[25] **B. P. Abbott, LIGO Scientific Collaboration and Virgo Collaboration, *GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence***, *American Physical Society, Phys. Rev. Lett. 116, 241103, 15th June 2016, doi:* https://doi.org/10.1103/PhysRevLett.116.241103.

[26] **Caltech and MIT, *"Gravitational Waves Detected 100 Years After Einstein's Prediction"***, *Web. Jan 2017,* https://www.ligo.caltech.edu/news/ligo20160211

[27] **California Institute of Technology and Massachusetts Institute of Technology, *LIGO Open Science Center***, *Web. Jan 2017, doi:* http://dx.doi.org/10.7935/K5MW2F23.

[28] **L. J. Rubbo, *Hands-on gravitational wave astronomy: Extracting astrophysical information from simulated signals***, *American Association of Physics Teachers, Am. J. Phys. 75, (2007); doi:* http://dx.doi.org/10.1119/1.2721587.

[29] **Caltech and MIT, *Abou aLigo***, *Web. Jan 2017,* https://www.ligo.caltech.edu/page/about-aligo.

[30] **Caltech and MIT, *LIGO Resumes Search for Gravitational Waves***, *Web. Jan 2017, http://www.ligo.caltech.edu/news/2016-11-30*.

[31] **NASA, *Mass Chart for Dead Stars and Black Holes***, *Web. Jan 2017, https://www.nasa.gov/jpl/nustar/pia18842*.

[32] **M. Coleman Miller, *Intermediate-Mass Black Holes***, *International Journal of Modern Physics, D13, 1 (2004), doi: https://dx.doi.org/10.1142/S0218271804004426*.

[33] **M. Coleman Miller, *Gravitational Radiation from Intermediate-Mass Black Holes***, *The American Astronomical Society, The Astrophysical Journal, Volume 581, Number 1 (2002), doi: https://doi.org/10.1086/344156*.

[34] **M. Punturo, *The Einstein Telescope: a third generation gravitational wave observatory***, *Classical and Quantum Gravity, 27 (2010) 194002 (12pp), doi: https://doi.org/10.1088/0264-9381/27/19/194002*.